

# Complex marginal deformations of D3-brane geometries, their Penrose limits and giant gravitons

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## Abstract

We apply the Lunin–Maldacena construction of gravity duals to  $\beta$ –deformed gauge theories to a class of Type IIB backgrounds with  $U(1)^3$  global symmetry, which include the multicenter D3-brane backgrounds dual to the Coulomb branch of  $\mathcal{N} = 4$  super Yang–Mills and the rotating D3-brane backgrounds dual to the theory at finite temperature and chemical potential. After a general discussion, we present the full form of the deformed metrics for three special cases, which can be used for the study of various aspects of the marginally-deformed gauge theories. We also construct the Penrose limits of the solutions dual to the Coulomb branch along a certain set of geodesics and, for the resulting PP–wave metrics, we examine the effect of  $\beta$ –deformations on the giant graviton states. We find that giant gravitons exist only up to a critical value of the  $\sigma$ –deformation parameter, are not degenerate in energy with the point graviton, and remain perturbatively stable. Finally, we probe the  $\sigma$ –deformed multicenter solutions by examining the static heavy-quark potential by means of Wilson loops. We find situations that give rise to complete screening as well as linear confinement, with the latter arising in an intriguing way reminiscent of phase transitions in statistical systems.

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# 1 Introduction

The AdS/CFT correspondence [1] has proven to be an invaluable tool for exploring the dynamics of large  $N$  gauge theories at strong coupling. In its original form, it relates  $\mathcal{N} = 4$ ,  $SU(N)$  super Yang-Mills theory to Type IIB string theory on  $\text{AdS}_5 \times \text{S}^5$ , with the limit of large 't Hooft coupling in the gauge theory corresponding to the classical supergravity limit of the string theory. The AdS/CFT correspondence can be extended to less symmetric theories, a class of which are the exactly marginal deformations of  $\mathcal{N} = 4$  SYM, introduced by Leigh and Strassler [2], which break supersymmetry down to  $\mathcal{N} = 1$ . The gravity duals of such deformations have been identified by Lunin and Maldacena [3] and are constructed by applying an  $SL(3, \mathbb{R})$  transformation or, equivalently [4], a certain sequence of T-dualities, S-dualities and coordinate shifts to the initial  $\text{AdS}_5 \times \text{S}^5$  solution. This construction has been generalized in [4, 5, 6] and extended to other backgrounds in [7] while diverse aspects of the deformation have been examined in [8, 9, 10, 11, 12].

The Lunin–Maldacena construction can be carried over to the Coulomb branch of the gauge theory. The latter is obtained by moving away from the conformal point at the origin of moduli space by giving nonzero *vevs* to the  $SO(6)$  scalars. The corresponding gravity duals are obtained by generalizing the stacked-brane distribution to a multicenter one, thereby breaking the  $SO(6)$  isometry of the solutions [13]. A class of marginal deformations ( $\gamma$ -deformations) of these solutions have been obtained by the procedure outlined above in [14]. Probes of the resulting deformed geometries with Wilson loops, according to the recipe of [15], have revealed a rich structure of phenomena in the gauge theory, with behaviors ranging from the standard Coulombic interaction to complete screening and linear or logarithmic confinement, while the wave equation for the radial modes of massless scalar excitations in the deformed backgrounds turns out to be related to the Inozemtsev  $BC_1$  integrable system [14]. Another class of marginal deformations ( $\sigma$ -deformations) of these solutions have been obtained in [16], where Wilson-loop calculations indicate the existence of a linear confining potential in some cases.

A further step forward would be to extend this construction to include the full set of complex  $\beta$ -deformations and to apply it for the most general case of non-extremal rotating D3-branes [13, 17, 18] which are dual to the gauge theory at finite temperature and R-charge chemical potentials and which include the multicenter D3-branes dual to the Coulomb branch as a limiting case. The construction of these deformed Type IIB backgrounds is the main purpose of this paper. After constructing the deformed solutions, we

explore diverse aspects of these backgrounds, namely the Penrose limits of the multicenter solutions, the giant graviton states supported in the resulting PP-waves and, finally, the Wilson-loop heavy-quark potential of the dual gauge theory. These investigations are carried out with emphasis on  $\sigma$ -deformations, as their effect is often overlooked in the literature although it is in many cases significant.

This paper is organized as follows: In section 2 we present in detail the  $\beta$ -deformation procedure for the most general Type IIB background consisting of the metric, a 4-form potential and a dilaton and possessing at least a  $U(1)^3$  global symmetry. In section 3 we consider a class of such backgrounds, corresponding to rotating and multicenter D3-branes, we specialize to three simple cases for which we present the explicit form of the deformed metrics, and we demonstrate the expected equivalence of the thermodynamics of the deformed and undeformed metrics. In section 4 we construct the PP-wave backgrounds arising as Penrose limits of the deformed multicenter solutions along a certain set of BPS and non-BPS geodesics. In section 5, we consider the simplest PP-wave background of this type and we investigate the effect of  $\sigma$ -deformations on the energetics of giant gravitons supported by this geometry. In section 6 we probe the deformed multicenter geometries by static Wilson loops for the case of  $\sigma$ -deformations. Finally, in section 7 we summarize and conclude. Our conventions for T- and S-duality are summarized in the appendix.

## 2 Marginal Deformations of Type IIB backgrounds with $U(1)^3$ isometry

### 2.1 Marginally deformed $\mathcal{N} = 4$ SYM and its gravity dual

On the gauge-theory side, our general setup refers to a class of exactly marginal deformations of  $\mathcal{N} = 4$  super Yang-Mills theory, namely the Leigh–Strassler  $\beta$ -deformations of [2]. In this construction, one starts from the  $\mathcal{N} = 4$  theory with complexified gauge coupling

$$\tau = \frac{\vartheta_{\text{YM}}}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2} , \quad (2.1)$$

and applies a deformation that acts on the three complex chiral superfields  $\Phi_i$ ,  $i = 1, 2, 3$ , of the theory by modifying the standard superpotential  $W = \text{Tr}(\Phi_1[\Phi_2, \Phi_3])$  to

$$W = \text{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2) , \quad (2.2)$$

where  $\beta$  is a complex phase. The latter is conveniently parametrized as  $\beta = \gamma - \tau\sigma$ , where  $\gamma$  and  $\sigma$  are real parameters with unit period; in the special cases  $\sigma = 0$  or  $\gamma = 0$ , the deformation is referred to as a  $\gamma$ -deformation or a  $\sigma$ -deformation respectively. The above deformation breaks  $\mathcal{N} = 4$  supersymmetry down to  $\mathcal{N} = 1$  and the corresponding  $SO(6)_{\mathcal{R}}$  global  $\mathcal{R}$ -symmetry group down to its  $U(1)_1 \times U(1)_2 \times U(1)_{\mathcal{R}}$  Cartan subgroup where  $U(1)_{\mathcal{R}}$  stands for the surviving R-symmetry. There is also a  $\mathbb{Z}_3$  symmetry under cyclic permutations. Under  $U(1)_1 \times U(1)_2$ , the charges of the chiral superfields are

$$(Q_1^{\Phi_1}, Q_1^{\Phi_2}, Q_1^{\Phi_3}) = (0, 1, -1) , \quad (Q_2^{\Phi_1}, Q_2^{\Phi_2}, Q_2^{\Phi_3}) = (-1, 1, 0) , \quad (2.3)$$

while the superpotential is invariant. The Coulomb branch of the theory is described by the F-term conditions

$$\Phi_1 \Phi_2 = q \Phi_2 \Phi_1 , \quad q \equiv e^{-2i\pi\beta} , \quad \text{and cyclic} , \quad (2.4)$$

which are valid for large  $N$  (exact for  $U(N)$ ). For generic  $\beta$ , these conditions are solved by traceless  $N \times N$  matrices, where in each entry at most one of them is nonzero. For  $\gamma$ -deformations with rational  $\gamma$ , the Coulomb branch contains additional regions [10, 11].

On the gravity side, the dual to the Leigh-Strassler deformation was constructed by Lunin and Maldacena in [3] for field theories possessing at least a  $U(1)_1 \times U(1)_2$  global symmetry which, in the gravity dual, corresponds to an isometry of the supergravity background along two angular directions parametrizing a 2-torus. The deformation proceeds by applying a certain  $\beta$ -dependent  $SL(3, \mathbb{R})$  transformation belonging to the first factor of the full  $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$  duality group of Type IIB supergravity on the 2-torus. The effect of this transformation in the field theory amounts to the replacement of the standard product of field operators by the Moyal-like product

$$f * g = e^{i\pi\beta(Q_1^f Q_2^g - Q_1^g Q_2^f)} f g , \quad (2.5)$$

which, for the case of  $\mathcal{N} = 4$  SYM, does indeed induce the modified superpotential (2.2). The transformation outlined above was applied to various Type IIB backgrounds in [3] and further generalized to a broader class of backgrounds in [7].

To illustrate the construction for the solutions of interest, we consider a general Type IIB background where the ten spacetime coordinates are split into a seven-dimensional part parametrized by  $x^I$ ,  $I = 1, 2, \dots, 7$  and a three-dimensional part corresponding to a 3-torus parametrized by the angles  $\phi_i$ ,  $i = 1, 2, 3$ . The most general metric of this form

is given by

$$ds_{10}^2 = G_{IJ}(x)dx^I dx^J + 2 \sum_{i=1}^3 \lambda_{Ii}(x)dx^I d\phi_i + \sum_{i=1}^3 z_i(x)d\phi_i^2 , \quad (2.6)$$

where the  $z_i$  are positive-definite functions. This metric contains the mixed  $dx^I d\phi_i$ -terms and generalizes that considered in [14] as the starting point for constructing marginally deformed backgrounds. Apart from the metric, the solution is characterized by the dilaton and the RR 4-form

$$A_4 = C_4 + C_3^i \wedge d\phi_i + \frac{1}{2}\epsilon_{ijk}C_2^i \wedge d\phi_j \wedge d\phi_k + C_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 , \quad (2.7)$$

where  $C_1$ ,  $C_2^i$ ,  $C_3^i$  and  $C_4$  are forms of degree indicated by the lower index. They have dependence and support only on  $x^I$  and are constrained by the self-duality relations

$$dC_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 = \star dC_4 , \quad \frac{1}{2}\epsilon_{ijk}dC_2^i \wedge d\phi_j \wedge d\phi_k = \star(dC_3^i \wedge d\phi_i) . \quad (2.8)$$

A special case of the above solution, which includes the rotating branes to be examined in Section 3, is the one where the only nonzero  $\lambda_{Ii}$  are those in the  $ti$  directions,  $\lambda_i \equiv \lambda_{ti}$ . For this case, the metric (2.6) simplifies to

$$ds_{10}^2 = G_{IJ}(x)dx^I dx^J + 2 \sum_{i=1}^3 \lambda_i(x)dt d\phi_i + \sum_{i=1}^3 z_i(x)d\phi_i^2 . \quad (2.9)$$

According to the gauge/gravity correspondence, the three chiral superfields  $\Phi_i$  are in one-to-one correspondence with three complex coordinates  $w_i = R_i(x)e^{i\phi_i}$  and the generators  $(Q_1, Q_2, Q_{\mathcal{R}})$  of the global symmetries of the gauge theory correspond to linear combinations of the generators  $(J_{\phi_1}, J_{\phi_2}, J_{\phi_3})$  of the shifts along the torus. To proceed, it is most convenient to trade the  $\phi_i$  for a new set of variables  $\varphi_i$  such that the generators  $(J_{\varphi_1}, J_{\varphi_2}, J_{\varphi_3})$  of shifts along these directions become precisely identified with  $(Q_1, Q_2, Q_{\mathcal{R}})$ . From (2.3), it is easily seen that the appropriate set of variables is

$$\varphi_1 = \frac{1}{3}(\phi_1 + \phi_2 - 2\phi_3) , \quad \varphi_2 = \frac{1}{3}(\phi_2 + \phi_3 - 2\phi_1) , \quad \varphi_3 = \frac{1}{3}(\phi_1 + \phi_2 + \phi_3) . \quad (2.10)$$

In terms of these new variables, the description of the  $SL(3, \mathbb{R})$  transformations corresponding to  $\beta$ -deformations in the gravity dual is rather simple. These transformations can be written [4] as the sequence  $STS^{-1}$  where  $T$  stands for a T-duality along the isometry direction  $\varphi_1$ ,  $S$  and  $S^{-1}$  stand for an S-duality with parameter  $\tilde{\sigma} \equiv \sigma/\gamma$  and  $-\tilde{\sigma}$  respectively (see the appendix for conventions), and  $s$  denotes the coordinate shift

$$s : \quad \varphi_2 \rightarrow \varphi_2 + \gamma\varphi_1 . \quad (2.11)$$

The special case of  $\gamma$ -deformations corresponds to the sequence  $TsT$  and is obtained from the above case by taking  $\tilde{\sigma} = 0$ . The detailed procedure is presented below.

## 2.2 Construction of the marginally-deformed solutions

We now construct the  $\beta$ -deformations of the gravity duals under consideration by applying the  $STS TS^{-1}$  sequence of transformations to the solution (2.6)–(2.8). Here, we will denote the metric components in the  $\phi$  and  $\varphi$  bases by  $G_{MN}$  and  $g_{MN}$  respectively, while we will indicate the various fields at the intermediate steps by superscripts, e.g.  $g_{MN}^{(1)}$ ,  $g_{MN}^{(2)}$ ,  $\dots$ , and those at the final step by a hat. In the various computations we use the T- and S-duality rules that we have written in the appendix in a compact way particularly useful for our purposes. To present the results in a succinct form, it is convenient to introduce the 2-forms

$$\begin{aligned}\mathcal{B}_2 &= \lambda_{I1}\lambda_{J2}dx^I \wedge dx^J + z_1(\lambda_{I2} - \lambda_{I3})d\phi_1 \wedge dx^I + z_1z_2d\phi_1 \wedge d\phi_2 + \text{cyclic} , \\ \mathcal{A}_2 &= C_1 \wedge (d\phi_1 + d\phi_2 + d\phi_3) + C_2^1 + C_2^2 + C_2^3 ,\end{aligned}\tag{2.12}$$

the first of which can be written as the product of two one-forms  $\mathcal{B}_2 = \mathcal{A}_1 \wedge \mathcal{B}_1$  with

$$\begin{aligned}\mathcal{A}_1 &= (\lambda_{I2} - \lambda_{I3})dx^I + z_2d\phi_2 - z_3d\phi_3 , \\ \mathcal{B}_1 &= \left( \frac{\lambda_{I2}z_3 + \lambda_{I3}z_2}{z_2 + z_3} - \lambda_{I1} \right) dx^I - z_1d\phi_1 + \frac{z_2z_3}{z_2 + z_3}(d\phi_2 + d\phi_3) ,\end{aligned}\tag{2.13}$$

and hence is nilpotent in the sense that

$$\mathcal{B}_2 \wedge \mathcal{B}_2 = 0 .\tag{2.14}$$

It is also useful to introduce

$$\begin{aligned}\mathcal{G}^{-1} &= 1 + |\beta|^2(z_1z_2 + z_2z_3 + z_3z_1) , \\ \mathcal{H} &= 1 + \sigma^2 e^{-2\Phi}(z_1z_2 + z_2z_3 + z_3z_1) , \\ \mathcal{Q} &= \gamma\sigma e^{-\Phi}(z_1z_2 + z_2z_3 + z_3z_1) ,\end{aligned}\tag{2.15}$$

where

$$\beta = \gamma - \tau\sigma = \gamma - i\sigma e^{-\Phi} ,\tag{2.16}$$

since the axion in the initial solution is zero. The calculation proceeds as follows:

- In the first step we perform an S-duality with parameter  $\tilde{\sigma} = \sigma/\gamma$ . Applying the

transformation and passing to the  $\varphi$  basis, we obtain the metric

$$\begin{aligned}
g_{IJ}^{(1)} &= \frac{|\beta|}{\gamma} G_{IJ}, \\
g_{I1}^{(1)} &= \frac{|\beta|}{\gamma} (\lambda_{I2} - \lambda_{I3}), \quad g_{I2}^{(1)} = \frac{|\beta|}{\gamma} (\lambda_{I2} - \lambda_{I1}), \quad g_{I3}^{(1)} = \frac{|\beta|}{\gamma} (\lambda_{I1} + \lambda_{I2} + \lambda_{I3}), \\
g_{11}^{(1)} &= \frac{|\beta|}{\gamma} (z_2 + z_3), \quad g_{22}^{(1)} = \frac{|\beta|}{\gamma} (z_1 + z_2), \quad g_{33}^{(1)} = \frac{|\beta|}{\gamma} (z_1 + z_2 + z_3), \\
g_{12}^{(1)} &= \frac{|\beta|}{\gamma} z_2, \quad g_{13}^{(1)} = \frac{|\beta|}{\gamma} (z_2 - z_3), \quad g_{23}^{(1)} = \frac{|\beta|}{\gamma} (z_2 - z_1),
\end{aligned} \tag{2.17}$$

the dilaton

$$e^{2\Phi^{(1)}} = \frac{|\beta|^4}{\gamma^4} e^{2\Phi}, \tag{2.18}$$

and the axion

$$A_0^{(1)} = -\frac{\gamma\sigma}{|\beta|^2} e^{-2\Phi}. \tag{2.19}$$

The RR 4-form remains unchanged and, in the new basis, reads

$$\begin{aligned}
A_4^{(1)} &= \mathcal{A}_2 \wedge d\varphi_1 \wedge d\varphi_2 + (C_2^1 + C_2^2 - 2C_2^3) \wedge d\varphi_2 \wedge d\varphi_3 + (C_2^2 + C_2^3 - 2C_2^1) \wedge d\varphi_3 \wedge d\varphi_1 \\
&+ (C_3^2 - C_3^3) \wedge d\varphi_1 + (C_3^2 - C_3^1) \wedge d\varphi_2 + (C_3^1 + C_3^2 + C_3^3) \wedge d\varphi_3 + C_4,
\end{aligned} \tag{2.20}$$

while the NSNS and RR 2-forms are zero.

- In the second step we perform a T-duality along  $\varphi_1$ . The NSNS fields are given by

$$\begin{aligned}
g_{IJ}^{(2)} &= \frac{|\beta|}{\gamma} \left( G_{IJ} - \frac{(\lambda_{I2} - \lambda_{I3})(\lambda_{J2} - \lambda_{J3})}{z_2 + z_3} \right), \\
g_{I2}^{(2)} &= \frac{|\beta|}{\gamma} \left( \frac{z_2\lambda_{I3} + z_3\lambda_{I2}}{z_2 + z_3} - \lambda_{I1} \right), \quad g_{I3}^{(2)} = \frac{|\beta|}{\gamma} \left( \frac{2(z_2\lambda_{I3} + z_3\lambda_{I2})}{z_2 + z_3} + \lambda_{I1} \right), \\
g_{11}^{(2)} &= \frac{\gamma}{|\beta|} \frac{1}{z_2 + z_3}, \quad g_{22}^{(2)} = \frac{|\beta|}{\gamma} \frac{z_1z_2 + z_2z_3 + z_3z_1}{z_2 + z_3}, \quad g_{33}^{(2)} = \frac{|\beta|}{\gamma} \left( z_1 + \frac{4z_2z_3}{z_2 + z_3} \right), \\
g_{23}^{(2)} &= \frac{|\beta|}{\gamma} \left( \frac{2z_2z_3}{z_2 + z_3} - z_1 \right), \\
B_2^{(2)} &= -\frac{1}{z_2 + z_3} [(\lambda_{I2} - \lambda_{I3})dx^I + z_2d\varphi_2 + (z_2 - z_3)d\varphi_3] \wedge d\varphi_1, \\
e^{2\Phi^{(2)}} &= \frac{|\beta|^3}{\gamma^3} \frac{e^{2\Phi}}{z_2 + z_3},
\end{aligned} \tag{2.21}$$

while the RR fields are

$$\begin{aligned}
A_1^{(2)} &= -\frac{\gamma\sigma}{|\beta|^2} e^{-2\Phi} d\varphi_1, \\
A_3^{(2)} &= -\mathcal{A}_2 \wedge d\varphi_2 + (C_2^2 + C_2^3 - 2C_2^1) \wedge d\varphi_3 + (C_3^2 - C_3^3), \\
A_5^{(2)} &= A_4^{(1)} \wedge d\varphi_1 + B_2^{(2)} \wedge A_3^{(2)}.
\end{aligned} \tag{2.22}$$

- In the third step we perform a coordinate shift  $\varphi_2 \rightarrow \varphi_2 + \gamma\varphi_1$ . The changed metric components are

$$\begin{aligned}
g_{I1}^{(3)} &= |\beta| \left( \frac{z_2\lambda_{I3} + z_3\lambda_{I2}}{z_2 + z_3} - \lambda_{I1} \right) , \\
g_{11}^{(3)} &= \frac{\gamma}{|\beta|} \mathcal{G}^{-1} \frac{1}{z_2 + z_3} , \\
g_{12}^{(3)} &= |\beta| \frac{z_1z_2 + z_2z_3 + z_3z_1}{z_2 + z_3} , \quad g_{13}^{(3)} = |\beta| \left( \frac{2z_2z_3}{z_2 + z_3} - z_1 \right) .
\end{aligned} \tag{2.23}$$

The 3-form changes as

$$A_3^{(3)} = A_3^{(2)} - \gamma \mathcal{A}_2 \wedge d\varphi_1 , \tag{2.24}$$

while all other fields remain invariant.

- In the fourth step another T-duality along  $\varphi_1$  is performed. Applying the duality and returning to the  $\phi$  basis, we find that the NSNS fields are given by

$$\begin{aligned}
G_{IJ}^{(4)} &= \frac{|\beta|}{\gamma} \{ G_{IJ} - |\beta|^2 \mathcal{G} [z_1(\lambda_{I2} - \lambda_{I3})(\lambda_{J2} - \lambda_{J3}) + \text{cyclic}] \} , \\
G_{Ii}^{(4)} &= \frac{|\beta|}{\gamma} \mathcal{G} [\lambda_{Ii} + |\beta|^2 (z_1z_2\lambda_{I3} + \text{cyclic})] , \\
G_{ij}^{(4)} &= \frac{|\beta|}{\gamma} \mathcal{G} (z_i\delta_{ij} + |\beta|^2 z_1z_2z_3) , \\
B_2^{(4)} &= \frac{|\beta|^2}{\gamma} \mathcal{G} \mathcal{B}_2 , \\
e^{2\Phi^{(4)}} &= \frac{|\beta|^4}{\gamma^4} \mathcal{G} e^{2\Phi} ,
\end{aligned} \tag{2.25}$$

while the RR fields are

$$\begin{aligned}
A_0^{(4)} &= -\frac{\gamma\sigma}{|\beta|^2} e^{-2\Phi} , \quad A_2^{(4)} = -\gamma \mathcal{A}_2 - \sigma e^{-2\Phi} \mathcal{G} \mathcal{B}_2 , \\
A_4^{(4)} &= A_4 - |\beta|^2 \mathcal{G} \mathcal{B}_2 \wedge \mathcal{A}_2 , \quad A_6^{(4)} = \frac{|\beta|^2}{\gamma} \mathcal{G} A_4 \wedge \mathcal{B}_2 .
\end{aligned} \tag{2.26}$$

- The final step is another S-duality, now with parameter  $-\tilde{\sigma}$ . This leads to the gravity

dual of the  $\beta$ -deformed theory, expressed in terms of the NSNS fields

$$\begin{aligned}
\hat{G}_{IJ} &= \mathcal{H}^{1/2} \{ G_{IJ} - |\beta|^2 \mathcal{G} [z_1(\lambda_{I2} - \lambda_{I3})(\lambda_{J2} - \lambda_{J3}) + \text{cyclic}] \} , \\
\hat{G}_{Ii} &= \mathcal{G} \mathcal{H}^{1/2} [\lambda_{Ii} + |\beta|^2 (z_1 z_2 \lambda_{I3} + \text{cyclic})] , \\
\hat{G}_{ij} &= \mathcal{G} \mathcal{H}^{1/2} (z_i \delta_{ij} + |\beta|^2 z_1 z_2 z_3) , \\
\hat{B}_2 &= \gamma \mathcal{G} \mathcal{B}_2 - \sigma \mathcal{A}_2 , \\
e^{2\Phi} &= \mathcal{G} \mathcal{H}^2 e^{2\Phi} ,
\end{aligned} \tag{2.27}$$

and the RR fields

$$\begin{aligned}
\hat{A}_0 &= \mathcal{H}^{-1} \mathcal{Q} e^{-\Phi} , \\
\hat{A}_2 &= -\gamma \mathcal{A}_2 - \sigma e^{-2\Phi} \mathcal{G} \mathcal{B}_2 , \\
\hat{A}_4 &= A_4 - \gamma^2 \mathcal{G} \mathcal{B}_2 \wedge \mathcal{A}_2 + \frac{1}{2} \gamma \sigma \mathcal{A}_2 \wedge \mathcal{A}_2 , \\
\hat{A}_6 &= \hat{B}_2 \wedge \hat{A}_4 ,
\end{aligned} \tag{2.28}$$

where in the last relation we made use of the nilpotency of  $\mathcal{B}_2$  (2.14). One can check that the above formulas reduce, for the appropriate limiting cases, to the various solutions that have been found in the literature [3, 4, 14]. The case of  $\gamma$ -deformations is obtained as the special case of the above where  $\sigma = 0$ , in which case the quantities in Eq. (2.15) reduce to

$$\mathcal{G}^{-1} = 1 + \gamma^2 (z_1 z_2 + z_2 z_3 + z_3 z_1) , \quad \mathcal{H} = 1 , \quad \mathcal{Q} = 0 . \tag{2.29}$$

At this point, it is instructive to compare the general case of  $\beta$ -deformations with the special case of  $\gamma$ -deformations. We see that the extra  $\sigma$ -dependence in the former enters through (i) the replacement  $\gamma^2 \rightarrow |\beta|^2$  in the metric and in the definition of  $\mathcal{G}$ , (ii) the overall factors  $\mathcal{H}^{1/2}$  and  $\mathcal{H}^2$  in the metric and dilaton respectively and (iii) a nonzero axion as well as new terms in the NSNS 2-form, the RR 2-form and the RR 4-form proportional to  $\mathcal{A}_2$ ,  $\mathcal{B}_2$  and  $\mathcal{A}_2 \wedge \mathcal{A}_2$ , respectively. These changes clearly affect the Nambu–Goto (or Dirac–Born–Infeld plus Wess–Zumino) actions of probe strings (or branes) propagating in the deformed geometry and so, in relevant investigations, one is entitled to expect qualitative departures from results obtained for purely  $\gamma$ -deformed backgrounds. On the other hand, one readily verifies that the massless scalar wave equation  $\partial_M (\sqrt{-G} e^{-2\Phi} G^{MN} \partial_N \Psi) = 0$  is insensitive to the presence of the  $\mathcal{H}$  factors, which implies that its analysis proceeds as in the  $\gamma$ -deformed case with the replacement

$\gamma^2 \rightarrow |\beta|^2$  and that, in the case of multicenter D3-branes considered in [14], its relation with integrable systems found there remains intact.

### 3 Deformations of rotating and multicenter D3-branes

In this section, we explicitly apply the  $\beta$ -deformation procedure described above to rotating D3-brane solutions. First, we give a brief review of the field-theory limit of these solutions and we focus on three special cases where the metrics simplify considerably. Then, we present the full metrics of the corresponding deformed solutions. Finally, we demonstrate that the thermodynamic properties of the deformed metrics are exactly the same as for the undeformed ones.

#### 3.1 Rotating and multicenter D3-brane solutions

The solutions we are interested in here are the non-extremal rotating D3-branes found in full generality in [13] using previous results from [17]. They are characterized by the non-extremality parameter  $\mu$  plus the rotation parameters  $a_i$ ,  $i = 1, 2, 3$ , which correspond to the three generators in the Cartan subalgebra of  $SO(6)$ . The spacetime coordinates are split into the brane coordinates  $(t, \vec{x}_3) = (t, x_1, x_2, x_3)$  and the transverse coordinates  $y_m$ ,  $m = 1, \dots, 6$ , which can be parametrized as

$$\begin{aligned} w_1 &= y_1 + iy_2 = \sqrt{r^2 + a_1^2} \sin \theta e^{i\phi_1}, \\ w_2 &= y_3 + iy_4 = \sqrt{r^2 + a_2^2} \cos \theta \sin \psi e^{i\phi_2}, \\ w_3 &= y_5 + iy_6 = \sqrt{r^2 + a_3^2} \cos \theta \cos \psi e^{i\phi_3}, \end{aligned} \quad (3.1)$$

where the complex coordinates  $w_i$  are in one-to-one correspondence with the chiral superfields  $\Phi_i$  of the gauge theory. Here we are interested in the field-theory limit of these solutions which, in the most general non-extremal case, is given by the metric (we follow [18], in which the thermodynamic properties of the solution, presented below in some

special cases, were also computed)

$$\begin{aligned}
ds^2 = & H^{-1/2} \left[ - \left( 1 - \frac{\mu^4}{r^4 \Delta} \right) dt^2 + d\vec{x}_3^2 \right] + H^{1/2} \frac{r^6 \Delta}{f} dr^2 \\
& + H^{1/2} \left[ r^2 \Delta_1 d\theta^2 + r^2 \Delta_2 \cos^2 \theta d\psi^2 + 2(a_2^2 - a_3^2) \cos \theta \sin \theta \cos \psi \sin \psi d\theta d\psi \right. \quad (3.2) \\
& + (r^2 + a_1^2) \sin^2 \theta d\phi_1^2 + (r^2 + a_2^2) \cos^2 \theta \sin^2 \psi d\phi_2^2 + (r^2 + a_3^2) \cos^2 \theta \cos^2 \psi d\phi_3^2 \\
& \left. - 2 \frac{\mu^2}{R^2} dt (a_1 \sin^2 \theta d\phi_1 + a_2 \cos^2 \theta \sin^2 \psi d\phi_2 + a_3 \cos^2 \theta \cos^2 \psi d\phi_3) \right] ,
\end{aligned}$$

the constant dilaton

$$e^\Phi = e^{\Phi_0} = g_s , \quad (3.3)$$

and a 4-form potential of the form (2.8) with

$$\begin{aligned}
C_4 &= -\frac{H^{-1}}{g_s} dt \wedge dx_1 \wedge dx_2 \wedge dx_3 , \\
(C_3^1, C_3^2, C_3^3) &= -\frac{\mu^2}{R^2 g_s} (a_1 \sin^2 \theta, a_2 \cos^2 \theta \sin^2 \psi, a_3 \cos^2 \theta \cos^2 \psi) dx_1 \wedge dx_2 \wedge dx_3 , \quad (3.4)
\end{aligned}$$

and  $C_1$  and  $C_2^i$  specified by the duality relations (2.8). In the above, the various functions are given by

$$\begin{aligned}
H &= \frac{R^4}{r^4 \Delta} , \\
f &= (r^2 + a_1^2)(r^2 + a_2^2)(r^2 + a_3^2) - \mu^4 r^2 , \\
\Delta &= 1 + \frac{a_1^2}{r^2} \cos^2 \theta + \frac{a_2^2}{r^2} (\sin^2 \theta \sin^2 \psi + \cos^2 \psi) + \frac{a_3^2}{r^2} (\sin^2 \theta \cos^2 \psi + \sin^2 \psi) \\
&+ \frac{a_2^2 a_3^2}{r^4} \sin^2 \theta + \frac{a_1^2 a_3^2}{r^4} \cos^2 \theta \sin^2 \psi + \frac{a_1^2 a_2^2}{r^4} \cos^2 \theta \cos^2 \psi , \quad (3.5) \\
\Delta_1 &= 1 + \frac{a_1^2}{r^2} \cos^2 \theta + \frac{a_2^2}{r^2} \sin^2 \theta \sin^2 \psi + \frac{a_3^2}{r^2} \sin^2 \theta \cos^2 \psi , \\
\Delta_2 &= 1 + \frac{a_2^2}{r^2} \cos^2 \psi + \frac{a_3^2}{r^2} \sin^2 \psi .
\end{aligned}$$

In the notation of Section 2, the metric components  $z_i$  and  $\lambda_i$  are given by

$$\begin{aligned}
(z_1, z_2, z_3) &= \frac{R^2}{r^2 \Delta^{1/2}} ((r^2 + a_1^2) \sin^2 \theta, (r^2 + a_2^2) \cos^2 \theta \sin^2 \psi, (r^2 + a_3^2) \cos^2 \theta \cos^2 \psi) , \\
(\lambda_1, \lambda_2, \lambda_3) &= -\frac{\mu^2}{r^2 \Delta^{1/2}} (a_1 \sin^2 \theta, a_2 \cos^2 \theta \sin^2 \psi, a_3 \cos^2 \theta \cos^2 \psi) , \quad (3.6)
\end{aligned}$$

For  $\mu \neq 0$ , these solutions describe rotating branes and are dual to  $\mathcal{N} = 4$  SYM at finite temperature and R-charge chemical potentials, and the  $a_i$  parametrize the angular

velocities/momenta on the supergravity side and the chemical potentials/R-charges on the gauge theory-side. For  $\mu = 0$ , these solutions describe multicenter brane distributions dual to the Coulomb branch of  $\mathcal{N} = 4$  SYM, and the  $a_i$  parametrize the principal radii of the distribution on the supergravity side and the scalar *vevs* on the gauge-theory side. In the rest of the paper, we will refer to the  $a_i$  as “rotation parameters”, keeping in mind their different interpretations in these two cases.

In what follows, we examine some simple special cases of the above general solution, namely those corresponding to three equal nonzero rotation parameters, two equal nonzero rotation parameters and one nonzero rotation parameter.

### Three equal rotation parameters

The first special case we consider is the one where all three rotation parameters are equal to each other,  $a_1 = a_2 = a_3 = r_0$ . Employing the change of variable  $r^2 \rightarrow r^2 - r_0^2$  we write the resulting metric as

$$ds^2 = H^{-1/2} \left[ - \left( 1 - \frac{\mu^4}{r^4} \right) dt^2 + d\vec{x}_3^2 \right] + H^{1/2} \frac{r^6}{r^6 - \mu^4(r^2 - r_0^2)} dr^2 + H^{1/2} \left\{ r^2 d\Omega_5^2 - \frac{2\mu^2 r_0}{R^2} dt [\sin^2 \theta d\phi_1 + \cos^2 \theta (\sin^2 \psi d\phi_2 + \cos^2 \psi d\phi_3)] \right\} , \quad (3.7)$$

where

$$H = \frac{R^4}{r^4} , \quad (3.8)$$

and  $d\Omega_5^2$  is the  $S^5$  metric

$$d\Omega_5^2 = d\theta^2 + \sin^2 \theta d\phi_1^2 + \cos^2 \theta (d\psi^2 + \sin^2 \psi d\phi_2^2 + \cos^2 \psi d\phi_3^2) . \quad (3.9)$$

The horizon radius  $r_H$  for this metric is given by the largest root of the equation

$$r^6 - \mu^4(r^2 - r_0^2) = 0 , \quad (3.10)$$

while its Hawking temperature reads

$$T = \frac{r_H(2r_H^2 - 3r_0^2)}{2\pi R^2(r_H^2 - r_0^2)} . \quad (3.11)$$

For  $\mu = 0$  this background reduces to the  $AdS_5 \times S^5$  background obtained by stacked D3-branes at the origin.

## Two equal rotation parameters

The second special case is the one where two rotation parameters are set to the same nonzero value, which we may take as  $a_2 = a_3 = r_0$ . Employing again the change of variable  $r^2 \rightarrow r^2 - r_0^2$  we have the metric

$$\begin{aligned} ds^2 = & H^{-1/2} \left[ - \left( 1 - \frac{\mu^4 H}{R^4} \right) dt^2 + d\vec{x}_3^2 \right] + H^{1/2} \frac{r^4(r^2 - r_0^2 \cos^2 \theta)}{(r^4 - \mu^4)(r^2 - r_0^2)} dr^2 \\ & + H^{1/2} \left[ (r^2 - r_0^2 \cos^2 \theta) d\theta^2 + r^2 \cos^2 \theta d\Omega_3^2 + (r^2 - r_0^2) \sin^2 \theta d\phi_1^2 \right. \\ & \left. - 2 \frac{\mu^2 r_0}{R^2} dt \cos^2 \theta (\sin^2 \psi d\phi_2 + \cos^2 \psi d\phi_3) \right], \end{aligned} \quad (3.12)$$

where

$$H = \frac{R^4}{r^2(r^2 - r_0^2 \cos^2 \theta)}, \quad (3.13)$$

while  $d\Omega_3^2$  is the  $S^3$  metric

$$d\Omega_3^2 = d\psi^2 + \sin^2 \psi d\phi_2^2 + \cos^2 \psi d\phi_3^2. \quad (3.14)$$

Now, the horizon radius is simply

$$r_H = \mu, \quad (3.15)$$

and the Hawking temperature reads

$$T = \frac{\sqrt{\mu^2 - r_0^2}}{\pi R^2}. \quad (3.16)$$

For  $\mu = 0$  this background reduces to that obtained by a uniform distribution of D3-branes on a 3-sphere of radius  $r_0$ .

## One rotation parameter

The third special case is the one where there is only one nonzero rotation parameter, which we may take as  $a_1 = r_0$ . In this case, we have the metric

$$\begin{aligned} ds^2 = & H^{-1/2} \left[ - \left( 1 - \frac{\mu^4 H}{R^4} \right) dt^2 + d\vec{x}_3^2 \right] + H^{1/2} \frac{r^2(r^2 + r_0^2 \cos^2 \theta)}{r^4 + r_0^2 r^2 - \mu^4} dr^2 \\ & + H^{1/2} \left[ (r^2 + r_0^2 \cos^2 \theta) d\theta^2 + r^2 \cos^2 \theta d\Omega_3^2 + (r^2 + r_0^2) \sin^2 \theta d\phi_1^2 \right. \\ & \left. - 2 \frac{\mu^2 r_0}{R^2} \sin^2 \theta dt d\phi_1 \right], \end{aligned} \quad (3.17)$$

where the harmonic function is given by

$$H = \frac{R^4}{r^2(r^2 + r_0^2 \cos^2 \theta)}, \quad (3.18)$$

while  $d\Omega_3^2$  is defined as before. The horizon radius is given by

$$r_H^2 = \frac{1}{2} \left( -r_0^2 + \sqrt{r_0^4 + 4\mu^4} \right) , \quad (3.19)$$

and the Hawking temperature reads

$$T = \frac{r_H \sqrt{r_0^4 + 4\mu^4}}{2\pi R^2 \mu^2} . \quad (3.20)$$

For  $\mu = 0$  this background reduces to that obtained by a uniform distribution of D3-branes on a disc of radius  $r_0$  and is related to the corresponding background for two rotation parameters by the transformation  $r_0^2 \rightarrow -r_0^2$ .

### 3.2 The deformed metrics

After the above preliminaries, we are ready to apply the marginal-deformation procedure to the rotating-brane solutions just discussed. In what follows, we present the explicit form of the marginally-deformed metrics for the three special cases considered earlier. To keep the notation as compact as possible, it is convenient to introduce the shorthand notation  $c_\alpha \equiv \cos \alpha$  and  $s_\alpha \equiv \sin \alpha$  and the following rescaling for the deformation parameters<sup>1</sup>

$$\hat{\beta} \equiv \frac{R^2 \beta}{2} , \quad \hat{\gamma} \equiv \frac{R^2 \gamma}{2} , \quad \hat{\sigma} \equiv \frac{R^2 \sigma}{2g_s} , \quad (3.21)$$

with the new parameters satisfying  $|\hat{\beta}|^2 = \hat{\gamma}^2 + \hat{\sigma}^2$ .

#### The zero-rotation case

As a first example, let us consider the case where all rotation parameters are set to zero. Setting  $r_0 = 0$  in any of the above three metrics and substituting in (2.26), we find

$$ds^2 = \mathcal{H}^{1/2} \left\{ H^{-1/2} \left[ - \left( 1 - \frac{\mu^4}{r^4} \right) dt^2 + d\vec{x}_3^2 \right] + H^{1/2} \left( \frac{r^4}{r^4 - \mu^4} dr^2 + r^2 d\Omega_{5,\beta}^2 \right) \right\} , \quad (3.22)$$

where  $d\Omega_{5,\beta}^2$  is the metric on the deformed five-sphere  $S_\beta^5$ , given by

$$d\Omega_{5,\beta}^2 = d\theta^2 + \mathcal{G} s_\theta^2 d\phi_1^2 + c_\theta^2 [d\psi^2 + \mathcal{G} (s_\psi^2 d\phi_2^2 + c_\psi^2 d\phi_3^2)] + \mathcal{G} |\hat{\beta}|^2 c_\theta^4 s_\theta^2 s_{2\psi}^2 \left( \sum_{i=1}^3 d\phi_i \right)^2 , \quad (3.23)$$

and the various functions are

$$\mathcal{G}^{-1} = 1 + 4|\hat{\beta}|^2 c_\theta^2 (s_\theta^2 + c_\theta^2 c_\psi^2 s_\psi^2) , \quad \mathcal{H} = 1 + 4\hat{\sigma}^2 c_\theta^2 (s_\theta^2 + c_\theta^2 c_\psi^2 s_\psi^2) , \quad (3.24)$$

and  $H = R^4/r^4$ . The resulting space is thus a conformal rescaling of the product of AdS<sub>5</sub>–Schwarzschild with  $S_\beta^5$ .

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<sup>1</sup>These parameters differ by the analogous parameters in [14] by a factor of 1/2.

### Three equal rotation parameters

Starting from the case with three equal rotation parameters, the deformed metric is found by substituting the undeformed metric (3.7) into Eq. (2.26). After some algebra, we find

$$\begin{aligned}
ds^2 = & \mathcal{H}^{1/2} H^{-1/2} \left\{ - \left[ 1 - \frac{\mu^4 [r^2 - |\hat{\beta}|^2 \mathcal{G} r_0^2 c_\theta^2 (c_\theta^4 s_{2\psi}^2 + 4s_\theta^4 + 4c_\theta^2 s_\theta^2 c_{4\psi})]}{r^6} \right] dt^2 + d\bar{x}_3^2 \right\} \\
& + \mathcal{H}^{1/2} H^{1/2} \left[ \frac{r^6}{r^6 - \mu^4 (r^2 - r_0^2)} dr^2 + r^2 d\Omega_{5,\beta}^2 \right] \\
& - \mathcal{G} \mathcal{H}^{1/2} H^{1/2} \frac{2\mu^2 r_0}{R^2} dt \left[ s_\theta^2 d\phi_1 + c_\theta^2 (s_\psi^2 d\phi_2 + c_\psi^2 d\phi_3) + 3|\hat{\beta}|^2 c_\theta^4 s_\theta^2 s_{2\psi}^2 \sum_{i=1}^3 d\phi_i \right], \tag{3.25}
\end{aligned}$$

where  $d\Omega_{5,\beta}^2$  is as in (3.23),  $\mathcal{G}^{-1}$  and  $\mathcal{H}$  are as in (3.24) and  $H$  is as in (3.8).

### Two equal rotation parameters

For the case with two equal rotation parameters, the deformed metric is found to be

$$\begin{aligned}
ds^2 = & \mathcal{H}^{1/2} H^{-1/2} \left\{ - \left[ 1 - \frac{\mu^4 \left[ 1 - \frac{|\hat{\beta}|^2 \mathcal{G} r_0^2 c_\theta^4 (4(r^2 - r_0^2) s_\theta^2 c_{2\psi}^2 + r^2 c_\theta^2 s_{2\psi}^2)}{r^2(r^2 - r_0^2 c_\theta^2)} \right]}{r^2(r^2 - r_0^2 c_\theta^2)} \right] dt^2 + d\bar{x}_3^2 \right\} \\
& + \mathcal{H}^{1/2} H^{1/2} \left[ \frac{r^4(r^2 - r_0^2 c_\theta^2)}{(r^2 - r_0^2)(r^4 - \mu^4)} dr^2 + (r^2 - r_0^2 c_\theta^2) d\theta^2 + r^2 c_\theta^2 d\psi^2 \right] \\
& + \mathcal{G} \mathcal{H}^{1/2} H^{1/2} \left\{ (r^2 - r_0^2) s_\theta^2 d\phi_1^2 + r^2 c_\theta^2 (s_\psi^2 d\phi_2^2 + c_\psi^2 d\phi_3^2) + \frac{|\hat{\beta}|^2 r^2 (r^2 - r_0^2) c_\theta^4 s_\theta^2 s_{2\psi}^2}{r^2 - r_0^2 c_\theta^2} \left( \sum_{i=1}^3 d\phi_i \right)^2 \right. \\
& \left. - \frac{2\mu^2 r_0}{R^2} dt \left[ c_\theta^2 (s_\psi^2 d\phi_2 + c_\psi^2 d\phi_3) + \frac{2|\hat{\beta}|^2 (r^2 - r_0^2) c_\theta^4 s_\theta^2 s_{2\psi}^2}{r^2 - r_0^2 c_\theta^2} \sum_{i=1}^3 d\phi_i \right] \right\}, \tag{3.26}
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{G}^{-1} &= 1 + 4|\hat{\beta}|^2 c_\theta^2 \frac{(r^2 - r_0^2) s_\theta^2 + r^2 c_\theta^2 c_\psi^2 s_\psi^2}{r^2 - r_0^2 c_\theta^2}, \\
\mathcal{H} &= 1 + 4\hat{\sigma}^2 c_\theta^2 \frac{(r^2 - r_0^2) s_\theta^2 + r^2 c_\theta^2 c_\psi^2 s_\psi^2}{r^2 - r_0^2 c_\theta^2}, \tag{3.27}
\end{aligned}$$

and  $H$  as in (3.13).

## One rotation parameter

For the case with one rotation parameter, the deformed metric is found to be

$$\begin{aligned}
ds^2 &= \mathcal{H}^{1/2} H^{-1/2} \left\{ - \left[ 1 - \frac{\mu^4(r^2 + r_0^2 c_\theta^2 - 4|\hat{\beta}|^2 \mathcal{G} r_0^2 c_\theta^2 s_\theta^4)}{r^2(r^2 + r_0^2 c_\theta^2)^2} \right] dt^2 + d\vec{x}_3^2 \right\} \\
&+ \mathcal{H}^{1/2} H^{1/2} \left[ \frac{r^2(r^2 + r_0^2 c_\theta^2)}{r^4 + r_0^2 r^2 - \mu^4} dr^2 + (r^2 + r_0^2 c_\theta^2) d\theta^2 + r^2 c_\theta^2 d\psi^2 \right] \\
&+ \mathcal{G} \mathcal{H}^{1/2} H^{1/2} \left[ (r^2 + r_0^2) s_\theta^2 d\phi_1^2 + r^2 c_\theta^2 (s_\psi^2 d\phi_2^2 + c_\psi^2 d\phi_3^2) + \frac{|\hat{\beta}|^2 r^2 (r^2 + r_0^2) c_\theta^4 s_\theta^2 s_{2\psi}^2}{r^2 + r_0^2 c_\theta^2} \left( \sum_{i=1}^3 d\phi_i \right)^2 \right. \\
&\left. - \frac{2\mu^2 r_0}{R^2} dt \left( s_\theta^2 d\phi_1 + \frac{|\hat{\beta}|^2 r^2 c_\theta^4 s_\theta^2 s_{2\psi}^2}{r^2 + r_0^2 c_\theta^2} \sum_{i=1}^3 d\phi_i \right) \right], \tag{3.28}
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{G}^{-1} &= 1 + 4|\hat{\beta}|^2 c_\theta^2 \frac{(r^2 + r_0^2) s_\theta^2 + r^2 c_\theta^2 c_\psi^2 s_\psi^2}{r^2 + r_0^2 c_\theta^2}, \\
\mathcal{H} &= 1 + 4\hat{\sigma}^2 c_\theta^2 \frac{(r^2 + r_0^2) s_\theta^2 + r^2 c_\theta^2 c_\psi^2 s_\psi^2}{r^2 + r_0^2 c_\theta^2}, \tag{3.29}
\end{aligned}$$

and  $H$  as in (3.18).

### 3.3 Invariance of the thermodynamics

A useful consistency check for our calculation is to examine the thermodynamic properties of the deformed rotating-brane solutions. In particular, since the deformed solutions are related to the undeformed ones by U-duality transformations that are symmetries of the underlying theory, all thermodynamic quantities for the deformed metrics must be equal to those for the undeformed ones. It is instructive to show that this is indeed the case by explicitly calculating the angular velocities, the Hawking temperature, and the entropy for the general deformed solutions.

We start by writing the deformation (2.27), restricted to a metric of the form (2.9), in the Einstein frame. In this frame, the deformed metric is  $\hat{G}_{MN}^{(E)} = e^{-\hat{\Phi}/2} \hat{G}_{MN} = g_s^{-1/2} \mathcal{G}^{-1/4} \mathcal{H}^{-1/2} \hat{G}_{MN}$ , and thus we have

$$\begin{aligned}
\hat{G}_{IJ}^{(E)} &= g_s^{-1/2} \mathcal{G}^{-1/4} \{ G_{IJ} - |\beta|^2 \mathcal{G} [z_1(\lambda_2 - \lambda_3)^2 + \text{cyclic}] \delta_{I,t} \delta_{J,t} \}, \\
\hat{G}_{ti}^{(E)} &= g_s^{-1/2} \mathcal{G}^{3/4} [\lambda_i + |\beta|^2 (z_1 z_2 \lambda_3 + \text{cyclic})], \\
\hat{G}_{ij}^{(E)} &= g_s^{-1/2} \mathcal{G}^{3/4} (z_i \delta_{ij} + |\beta|^2 z_1 z_2 z_3), \tag{3.30}
\end{aligned}$$

where we recall that, for the general rotating-brane solution (3.2),  $z_i$  and  $\lambda_i$  are given in (3.6) and  $\mathcal{G}$  and  $\mathcal{H}$  are given in terms of the  $z_i$  in (2.15). Letting  $G_{MN}^{(\text{E})} = e^{-\Phi_0/2} G_{MN} = g_s^{-1/2} G_{MN}$  be the undeformed metric in the Einstein frame, we write

$$\hat{G}_{MN}^{(\text{E})} = G_{MN}^{(\text{E})} + \delta G_{MN}^{(\text{E})}, \quad (3.31)$$

where the functions  $\delta G_{MN}^{(\text{E})}$  represent the effect of the deformation and are read off from (3.30) and the explicit relations (3.6) for  $z_i$  and  $\lambda_i$  and (2.15) for  $\mathcal{G}$  and  $\mathcal{H}$ . Although the resulting  $\delta G_{MN}^{(\text{E})}$  are very complicated functions of  $r$ ,  $\theta$  and  $\psi$ , inspection of (2.15) and (3.6) shows that they satisfy

$$\delta G_{MN}^{(\text{E})}|_{(\theta,\psi)=(\pi/2,0)} = \delta G_{MN}^{(\text{E})}|_{(\theta,\psi)=(0,\pi/2)} = \delta G_{MN}^{(\text{E})}|_{(\theta,\psi)=(0,0)} = 0, \quad (3.32)$$

while it can be shown that

$$\partial_{r,\theta,\psi} \delta G_{MN}^{(\text{E})}|_{(\theta,\psi)=(\pi/2,0)} = \partial_{r,\theta,\psi} \delta G_{MN}^{(\text{E})}|_{(\theta,\psi)=(0,\pi/2)} = \partial_{r,\theta,\psi} \delta G_{MN}^{(\text{E})}|_{(\theta,\psi)=(0,0)} = 0. \quad (3.33)$$

That is, there exist three values of  $(\theta, \psi)$  for which the metric and its derivatives reduce to those in the undeformed case.

Given Eqs. (3.32) and (3.33), it is very easy to check that the angular velocities and the Hawking temperature are the same as in the undeformed solution. Indeed, it immediately follows that the horizon radius for the deformed metric is equal to the horizon radius  $r_H$  for the undeformed one. The angular velocities  $\hat{\Omega}_i$ , are found by demanding that the Killing vector  $\xi = \partial_t + \hat{\Omega}_i \partial_{\phi_i}$  associated with a stationary observer be null at  $r = r_H$ , i.e. by solving the equation

$$\hat{\xi}^2(r_H) = \xi^2(r_H) + \delta G_{tt}^{(\text{E})}(r_H) + 2\delta G_{ti}^{(\text{E})}(r_H)\hat{\Omega}_i + \delta G_{ij}^{(\text{E})}(r_H)\hat{\Omega}_i\hat{\Omega}_j = 0, \quad (3.34)$$

where  $\xi^2$  and  $\hat{\xi}^2$  are the norms of  $\xi$  with the metrics  $G_{MN}^{(\text{E})}$  and  $\hat{G}_{MN}^{(\text{E})}$ , respectively. Evaluating this equation at  $(\theta, \psi) = (\pi/2, 0)$ ,  $(0, \pi/2)$  and  $(0, 0)$  and using (3.32), we obtain three decoupled equations for  $\hat{\Omega}_1$ ,  $\hat{\Omega}_2$  and  $\hat{\Omega}_3$ , respectively, which are the same ones that arise for the undeformed metric [18]. Therefore, the angular velocities are the same,  $\hat{\Omega}_i = \Omega_i$ . The Hawking temperature is found from the relation

$$\hat{T}_H^2 = \frac{1}{16\pi^2} \lim_{r \rightarrow r_H} \frac{\hat{G}^{(\text{E})MN} \partial_M \hat{\xi}^2 \partial_N \hat{\xi}^2}{-\hat{\xi}^2}, \quad (3.35)$$

which, being independent of the angles, can be evaluated at any of the aforementioned three values of  $(\theta, \psi)$ . Then, use of (3.32) and (3.33) leads to the same relation as for the undeformed metric and so  $\hat{T}_H = T_H$ .

Finally, the entropy is determined by the horizon area which is in turn related to the determinant of the eight-dimensional metric along the directions normal to the horizon. Labelling these directions as  $A = (\alpha, i)$  with  $\alpha = (\vec{x}_3, \theta, \psi)$  and  $i = (\phi_1, \phi_2, \phi_3)$ , we find that this eight-dimensional metric equals

$$G_{AB}^{(E)} = g_s^{-1/2} \begin{pmatrix} G_{\alpha\beta} & 0 \\ 0 & z_i \delta_{ij} \end{pmatrix} \quad (3.36)$$

and

$$\hat{G}_{AB}^{(E)} = g_s^{-1/2} \begin{pmatrix} \mathcal{G}^{-1/4} G_{\alpha\beta} & 0 \\ 0 & \mathcal{G}^{3/4} (z_i \delta_{ij} + |\beta|^2 z_1 z_2 z_3) \end{pmatrix}, \quad (3.37)$$

for the undeformed and deformed cases, respectively. Then, a simple calculation gives

$$\begin{aligned} \det \hat{G}_{AB}^{(E)} &= g_s^{-4} \mathcal{G} [1 + |\beta|^2 (z_1 z_2 + z_2 z_3 + z_3 z_1)] z_1 z_2 z_3 \det G_{\alpha\beta} \\ &= g_s^{-4} z_1 z_2 z_3 \det G_{\alpha\beta} = \det G_{AB}^{(E)}, \end{aligned} \quad (3.38)$$

where in the second line we used the defining relation for  $\mathcal{G}$ . Therefore the entropy is indeed invariant, a fact that seems to be closely related to the invariance of the central charge of the dual CFT under deformations [9]. This completes our consistency check.

## 4 Penrose limits of the $\beta$ -deformed solutions

Having constructed the deformed solutions, it is interesting to examine their Penrose limits, following by applying the standard procedure introduced by [19] to the metric and the other fields of the solution.<sup>2</sup> The resulting spacetimes are PP-waves that constitute generalizations of the maximally supersymmetric PP-wave solution [22] of Type IIB string theory and, in the context of the AdS/CFT correspondence, are related to the BMN limit [23, 24] of the gauge theory. PP-wave limits of marginally-deformed backgrounds were first considered in [25] and in [3], with the former construction starting from the gauge-theory side of the correspondence, while further aspects of such solutions were examined in [26]. Here, we further generalize these constructions to include the effects of  $\sigma$ -deformations and of turning on rotation parameters by following [27] where PP-wave solutions based on the solutions of subsection 3.1 were constructed and further analyzed.

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<sup>2</sup>In a string theory context, in the presence of non-vanishing scalar and tensor fields, the Penrose limiting procedure for constructing PP-wave solutions was first applied in [20, 21].

## 4.1 Null geodesics in the deformed geometry

To find the Penrose limits of the deformed solutions, we first need to identify null geodesics in the respective geometries. The geodesics we are interested in involve  $t$ ,  $r$  and one linear combination of the cyclic coordinates  $\phi_i$  which we denote by  $\phi$ , taking the remaining coordinates to constant values consistent with their equations of motion. To seek such geodesics, we note that the  $\phi_i$  are cyclic and hence setting any of them to any constant value is automatically consistent, while an ansatz with constant  $\theta$  and  $\psi$  is certainly consistent if  $\partial_\theta G_{ij} \dot{\phi}_i \dot{\phi}_j = \partial_\psi G_{ij} \dot{\phi}_i \dot{\phi}_j = 0$  and  $\partial_\theta G_{ti} \dot{\phi}_i = \partial_\psi G_{ti} \dot{\phi}_i = 0$ , where the dot denotes differentiation with respect to the affine parameter  $\tau$  of the geodesic. In our examples with nonzero rotation ( $r_0 \neq 0$ ), these equations are solved if

$$\begin{aligned} \theta &= \frac{\pi}{2}, & \psi, \phi_i &= \text{any}, \\ \theta &= 0, & \psi &= 0, \frac{\pi}{2}, & \phi_i &= \text{any}, \\ \theta &= 0, & \psi &= \frac{\pi}{4}, & \phi_2 &= \phi_3, & \phi_1 &= \text{any}. \end{aligned} \tag{4.1}$$

For the case of zero rotation ( $r_0 = 0$ ), there emerge additional solutions, one of which is

$$\theta = \arcsin \frac{1}{\sqrt{3}}, \quad \psi = \frac{\pi}{4}, \quad \phi_1 = \phi_2 = \phi_3. \tag{4.2}$$

Regarding the sensitivity of the metrics along these geodesics to  $\beta$ -deformations, we note that the latter are non-trivial only when  $z_1 z_2 + z_2 z_3 + z_3 z_1 \neq 0$  which, by (3.6), requires that  $\theta \neq \pi/2$  and  $(\theta, \psi) \neq (0, 0), (0, \pi/2)$ . Therefore, the effective metrics along the geodesics in the first two lines of (4.1) are insensitive to  $\beta$ -deformations while those along the geodesics in the third line of (4.1) and in (4.2) are sensitive to  $\beta$ -deformations. Setting the unspecified  $\phi_i$  to zero, we are led to consider the following cases

$$\begin{aligned} (J, 0, 0) : \quad \theta &= \frac{\pi}{2}, & \psi &= 0, & \phi_1 &= \phi, & \phi_2 &= \phi_3 = 0, \\ (0, J, 0) : \quad \theta &= 0, & \psi &= \frac{\pi}{2}, & \phi_2 &= \phi, & \phi_3 &= \phi_1 = 0, \\ (0, 0, J) : \quad \theta &= 0, & \psi &= 0, & \phi_3 &= \phi, & \phi_1 &= \phi_2 = 0, \\ (J, J, J) : \quad \theta &= \arcsin \frac{1}{\sqrt{3}}, & \psi &= \frac{\pi}{4}, & \varphi_3 &= \phi, & \varphi_1 = \varphi_2 &= 0; & \text{for } r_0 = 0, \\ (0, J, J) : \quad \theta &= 0, & \psi &= \frac{\pi}{4}, & \varphi &\equiv \frac{\phi_2 + \phi_3}{2} = \phi, & \chi &\equiv \frac{\phi_2 - \phi_3}{2} = 0, & \phi_1 &= 0, \end{aligned} \tag{4.3}$$

where, in the fourth line, the  $\varphi_i$  are as given in (2.10). The above cases correspond to a particle moving with angular momenta  $(J_{\phi_1}, J_{\phi_2}, J_{\phi_3}) = (J, 0, 0), (0, J, 0), (0, 0, J), (J, J, J)$  and  $(0, J, J)$  respectively along the three isometry directions.

To examine the properties of these geodesics in the various backgrounds, we distinguish the following cases:

- *Undeformed, zero rotation.* In this case, the five-sphere is round and the full  $SO(6)$  isometry group operates. All choices correspond to BPS geodesics that can be rotated into one another.
- *Deformed, zero rotation.* Here, the five-sphere is deformed, with the isometry group broken to  $U(1)^3$ . The choices  $(J, 0, 0)$ ,  $(0, J, 0)$  and  $(0, 0, J)$  correspond to three BPS geodesics that can still be rotated into each other, the choice  $(J, J, J)$  corresponds to a distinct BPS geodesic, and the choice  $(0, J, J)$  corresponds to a distinct non-BPS geodesic.
- *Deformed, nonzero rotation.* Now, the deformed five-sphere is in addition squashed. The available choices  $(J, 0, 0)$ ,  $(0, J, 0)$ ,  $(0, 0, J)$  and  $(0, J, J)$  are all generically inequivalent but, for the specific backgrounds considered in section 3.2, the choices  $(0, J, 0)$  and  $(0, 0, J)$  remain equivalent and it suffices to consider only one of them, say the second.

We note that the  $(J, J, J)$  geodesic has been first considered for undeformed  $AdS_5 \times S^5$  in [25] (see also [26]), which is also where Penrose limits of marginally-deformed  $AdS_5 \times S^5$  first appeared, found through field-theory considerations.

## 4.2 The Penrose limit

Having identified the geodesics of interest, we are ready to take the Penrose limit, proceeding along the lines of [27]. We first employ the rescaling  $(t, \vec{x}_3) \rightarrow R^2(t, \vec{x}_3)$  and we write the effective metric for  $t$ ,  $r$  and  $\phi$  as

$$\frac{ds_3^2}{R^2} = \gamma_{tt} dt^2 + \gamma_{rr} dr^2 + \gamma_{\phi\phi} d\phi^2 + 2\gamma_{t\phi} dt d\phi. \quad (4.4)$$

Independence of the metric from  $t$  and  $\phi$  leads to two conserved quantities, associated with the Killing vectors  $k = \partial_t$  and  $l = \partial_\phi$  and identified with the energy and the angular momentum, namely

$$E = -k^\mu u_\mu = -\gamma_{tt} \dot{t} - \gamma_{t\phi} \dot{\phi} = 1, \quad J = l^\mu u_\mu = \gamma_{t\phi} \dot{t} - \gamma_{\phi\phi} \dot{\phi}, \quad (4.5)$$

Solving for  $\dot{t}$  and  $\dot{\phi}$  and substituting into  $ds_3^2 = 0$ , we obtain the equation

$$\dot{r}^2 = \frac{\gamma_{\phi\phi} + 2J\gamma_{t\phi} + J^2\gamma_{tt}}{\gamma_{rr}(\gamma_{t\phi}^2 - \gamma_{tt}\gamma_{\phi\phi})}, \quad (4.6)$$

whose solution determines  $r$  in terms of  $\tau$ . We next change variables from  $(r, t, \phi)$  to the new variables  $(u, v, y)$ , defined according to

$$\begin{aligned} dr &= \sqrt{\frac{\gamma_{\phi\phi} + 2J\gamma_{t\phi} + J^2\gamma_{tt}}{\gamma_{rr}(\gamma_{t\phi}^2 - \gamma_{tt}\gamma_{\phi\phi})}} du , \\ dt &= \frac{\gamma_{\phi\phi} + J\gamma_{t\phi}}{\gamma_{t\phi}^2 - \gamma_{tt}\gamma_{\phi\phi}} du - \frac{1}{R^2} dv + \frac{J}{R} dx , \\ d\phi &= -\frac{\gamma_{t\phi} + J\gamma_{tt}}{\gamma_{t\phi}^2 - \gamma_{tt}\gamma_{\phi\phi}} du + \frac{1}{R} dx , \end{aligned} \quad (4.7)$$

so that, in particular,  $u$  is identified with the affine parameter  $\tau$ . We then rescale the spatial brane coordinates as

$$\vec{x}_3 = \frac{\vec{r}_3}{R} . \quad (4.8)$$

Finally, depending on the case at hand, we make the following changes of angular variables

$$\begin{aligned} (J, 0, 0) : \quad \theta &= \frac{\pi}{2} - \frac{\rho}{R} , \\ (0, 0, J) : \quad \theta &= \frac{\rho_1}{R} , \quad \psi = \frac{\rho_2}{R} , \\ (J, J, J) : \quad \theta &= \arcsin \frac{1}{\sqrt{3}} + \frac{y_1}{R} , \quad \psi = \frac{\pi}{4} + \frac{y_2}{R} , \quad \varphi_1 = -\frac{y_3 - \sqrt{3}y_4}{\sqrt{2}R} , \quad \varphi_2 = \frac{\sqrt{2}y_3}{R} , \\ (0, J, J) : \quad \theta &= \frac{\rho_1}{R} , \quad \psi = \frac{\pi}{4} + \frac{\rho}{R} , \quad \chi = \frac{\tilde{\rho}}{R} . \end{aligned} \quad (4.9)$$

The Penrose limit is then obtained by substituting all these changes of variables into the original solution and taking the limit  $R \rightarrow \infty$ . For the marginally-deformed metrics of interest, this limit must be taken while keeping  $\hat{\gamma} \sim R^2\gamma$  and  $\hat{\sigma} \sim R^2\sigma$  fixed. The resulting metric always takes the form of a PP-wave in Rosen-like coordinates and can be brought to Brinkmann-like coordinates by suitable changes of variables.

### 4.3 PP-wave limits of the deformed solutions

Here, we employ the method described above to derive the PP-wave limit of the marginally-deformed metrics of section 3.2. To keep things relatively simple, we consider only the case  $\mu = 0$ , corresponding to D3-brane distribution, in which case the ‘‘rotation parameter’’  $r_0$  is to be thought of as the radius of the D3-brane distribution. Note also that, although the  $(J, 0, 0)$  and  $(0, 0, J)$  geodesics are insensitive to the  $\beta$ -deformation, the PP-wave backgrounds resulting from the limiting procedure described above are sensitive to the deformation.

## The zero-rotation case

We begin with the zero-rotation case where the deformed metric is a conformal rescaling of  $\text{AdS}_5 \times S^5_\beta$ . The results for the various geodesics are as follows:

- **$(J, 0, 0)$  geodesic.** As a warmup exercise, and for later reference, we first review the construction of the simplest PP-wave limit of marginally-deformed  $\text{AdS}_5 \times S^5$ , first derived in [3], in some detail. The differential equation for  $r$  becomes

$$\dot{r}^2 = 1 - J^2 r^2 , \quad (4.10)$$

with solution

$$r^2(u) = \frac{1}{J^2} \sin^2 Ju . \quad (4.11)$$

After following the above limiting procedure we find that the Penrose limit of the deformed metric reads

$$ds^2 = 2dudv + A_r d\bar{r}_3^2 + A_x dx^2 + d\bar{y}_4^2 - C du^2 , \quad (4.12)$$

where  $\bar{y}_4$  is defined by

$$d\bar{y}_4^2 = d\rho^2 + \rho^2(d\psi^2 + \sin^2 \psi d\phi_2^2 + \cos^2 \psi d\phi_3^2) , \quad (4.13)$$

and the various functions are given by

$$A_r = r^2 = \frac{1}{J^2} \sin^2 Ju , \quad A_x = 1 - J^2 r^2 = \cos^2 Ju , \quad C = (1 + 4|\hat{\beta}|^2) J^2 \bar{y}_4^2 . \quad (4.14)$$

For future reference, we also write the Rosen form of the remaining nonzero fields, although we will not do so for the remaining cases considered below. We have

$$\begin{aligned} B_2 &= 2J\hat{\gamma}\rho^2 du \wedge (\sin^2 \psi d\phi_2 - \cos^2 \psi d\phi_3) , \\ e^{2\Phi} &= g_s^2 \\ A_2 &= -\frac{2J\hat{\sigma}}{g_s} \rho^2 du \wedge (\sin^2 \psi d\phi_2 - \cos^2 \psi d\phi_3) , \\ A_4 &= \frac{J}{g_s} \left( \frac{\sin^4 Ju}{J^4} dr_1 \wedge dr_2 \wedge dr_3 \wedge dx + \rho^4 \cos \psi \sin \psi du \wedge d\psi \wedge d\phi_2 \wedge d\phi_3 \right) . \end{aligned} \quad (4.15)$$

Applying standard transformations<sup>3</sup> to pass to Brinkmann coordinates, we write the metric as

$$ds^2 = 2dudv + d\bar{r}_4^2 + d\bar{y}_4^2 + (F_r \bar{r}_4^2 + F_y \bar{y}_4^2) du^2 , \quad (4.16)$$

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<sup>3</sup>For a metric in the Rosen form  $ds^2 = 2dudv + \sum_i A_i(u) dx_i^2 - C du^2$ , applying the sequence of coordinate transformations  $x_i \rightarrow \frac{x_i}{\sqrt{A_i}}$  and  $v \rightarrow v + \frac{1}{4} \sum_i \frac{d \ln A_i}{du} x_i^2$  brings it to the Brinkmann form  $ds^2 = 2dudv + \sum_i dx_i^2 + (\sum_i F_i x_i^2 - C) du^2$ , where  $F_i = \frac{1}{4} \left( \frac{d \ln A_i}{du} \right)^2 + \frac{1}{2} \frac{d^2 \ln A_i}{du^2}$ .

where  $\vec{r}_4 = (\vec{r}_3, x)$  and

$$F_r = -J^2, \quad F_y = -(1 + 4|\hat{\beta}|^2)J^2, \quad (4.17)$$

and the remaining nonzero fields of the solution as

$$\begin{aligned} H_3 &= -4J\hat{\gamma}du \wedge (dy_1 \wedge dy_2 - dy_3 \wedge dy_4), \\ e^{2\Phi} &= g_s^2 \\ F_3 &= \frac{4J\hat{\sigma}}{g_s}du \wedge (dy_1 \wedge dy_2 - dy_3 \wedge dy_4), \\ F_5 &= \frac{J}{g_s}du \wedge (dr_1 \wedge dr_2 \wedge dr_3 \wedge dr_4 - dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4). \end{aligned} \quad (4.18)$$

We see that the deformation affects only the function  $F_y$  in the metric and the NSNS and RR 3-form field strengths.

- **$(0, 0, J)$  geodesic.** The solution for  $r(u)$  is the same as before and the Penrose limit of the metric in Rosen coordinates is given by (4.12) and (4.14) where  $\vec{y}_4$  is now defined by

$$d\vec{y}_4^2 = (dy_1^2 + dy_2^2) + (dy_3^2 + dy_4^2) = (d\rho_1^2 + \rho_1^2 d\phi_1^2) + (d\rho_2^2 + \rho_2^2 d\phi_2^2), \quad (4.19)$$

In Brinkmann coordinates, the Penrose limit is given by (4.17) and (4.18). We explicitly verify that, for  $r_0 = 0$ , the Penrose limits for the  $(J, 0, 0)$  and  $(0, J, 0)$  geodesics are equivalent as already remarked in the comments following (4.3).

- **$(J, J, J)$  geodesic.** For this case, it is convenient to introduce

$$\mathcal{G}^{-1} = 1 + \frac{4|\hat{\beta}|^2}{3}, \quad \mathcal{H} = 1 + \frac{4\hat{\sigma}^2}{3}. \quad (4.20)$$

which are just the effective constant values of the functions in (3.24) for the given ansatz. Then, the differential equation for  $r$  has the form

$$\mathcal{H}\dot{r}^2 = 1 - J^2r^2, \quad (4.21)$$

with solution

$$r^2(u) = \frac{1}{J^2} \sin^2 \left( \frac{J}{\mathcal{H}^{1/2}} u \right). \quad (4.22)$$

In Rosen-like coordinates, the Penrose limit of the deformed metric reads

$$ds^2 = 2dudv + A_r dr_3^2 + A_x dx^2 + A_y dy_2^2 + \tilde{A}_y d\vec{y}_2^2 + B_y(y_1 dy_3 - y_2 dy_4)du - C du^2, \quad (4.23)$$

where

$$d\vec{y}_2^2 = dy_1^2 + dy_2^2, \quad d\vec{y}_2^2 = dy_3^2 + dy_4^2, \quad (4.24)$$

and

$$\begin{aligned}
A_r &= \mathcal{H}^{1/2} r^2 = \frac{\mathcal{H}^{1/2}}{J^2} \sin^2 \left( \frac{J}{\mathcal{H}^{1/2}} u \right) , \\
A_x &= 1 - J^2 r^2 = \cos^2 \left( \frac{J}{\mathcal{H}^{1/2}} u \right) \\
A_y &= \mathcal{H}^{1/2} , \quad \tilde{A}_y = \mathcal{G} \mathcal{H}^{1/2} , \quad B_y = -4\mathcal{G}J , \\
C &= \frac{16J^2 |\hat{\beta}|^2 \mathcal{G} \bar{y}_2^2}{3\mathcal{H}^{1/2}} .
\end{aligned} \tag{4.25}$$

In Brinkmann-like coordinates, the metric reads

$$ds^2 = 2dudv + d\bar{r}_4^2 + d\bar{y}_2^2 + d\bar{y}_2^2 + G_2(y_1 dy_3 - y_2 dy_4)du + (F_r \bar{r}_4^2 + F_y \bar{y}_2^2)du^2 , \tag{4.26}$$

where  $\bar{r}_4 = (\bar{r}_3, x)$  and

$$F_r = -\frac{J^2}{\mathcal{H}} , \quad F_y = -\frac{16J^2 |\hat{\beta}|^2 \mathcal{G}}{3\mathcal{H}} , \quad G_2 = -\frac{4\mathcal{G}^{1/2}}{\mathcal{H}^{1/2}} , \tag{4.27}$$

and the remaining fields of the solution are

$$\begin{aligned}
H_3 &= -\frac{4J}{\sqrt{3}\mathcal{H}} du \wedge [2\hat{\sigma}g_s dy_1 \wedge dy_2 + \hat{\gamma}\mathcal{G}^{1/2}(dy_1 \wedge dy_4 + dy_2 \wedge dy_3)] , \\
e^{2\Phi} &= \mathcal{G} \mathcal{H}^2 g_s^2 \\
F_3 &= -\frac{4J}{\sqrt{3}\mathcal{H}} du \wedge [2\hat{\gamma}dy_1 \wedge dy_2 - \frac{\hat{\sigma}}{g_s} \mathcal{G}^{1/2}(dy_1 \wedge dy_4 + dy_2 \wedge dy_3)] , \\
F_5 &= \frac{J}{\mathcal{H}^{3/2} g_s} du \wedge (dr_1 \wedge dr_2 \wedge dr_3 \wedge dr_4 - dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4) .
\end{aligned} \tag{4.28}$$

This is the generalization of the PP-wave considered in [25, 26] which includes the effect of  $\sigma$ -deformations. Now, the deformation affects all the  $F$ -functions in the metric and all nonzero fields. This type of PP-wave falls into the subclass of homogeneous plane waves considered in [28].

•  **$(0, J, J)$  geodesic.** For this case, we introduce

$$\mathcal{G}^{-1} = 1 + |\hat{\beta}|^2 , \quad \mathcal{H} = 1 + \hat{\sigma}^2 , \tag{4.29}$$

which are again the effective constant values of the functions in (3.24). Then, the differential equation for  $r$  has the form

$$\mathcal{H} \dot{r}^2 = 1 - \frac{J^2 r^2}{\mathcal{G}} , \tag{4.30}$$

with solution

$$r^2(u) = \frac{\mathcal{G}}{J^2} \sin^2 \left( \frac{J}{\mathcal{G}^{1/2} \mathcal{H}^{1/2}} u \right) . \tag{4.31}$$

In Rosen-like coordinates, the Penrose limit of the deformed metric reads

$$\begin{aligned} ds^2 = & 2dudv + A_r d\bar{r}_3^2 + A_x dx^2 + A_y d\bar{y}_2^2 + B_y (y_1 dy_2 - y_2 dy_1) du \\ & + A_\rho d\rho^2 + \tilde{A}_\rho d\tilde{\rho}^2 + B_\rho \rho d\tilde{\rho} du - C du^2 , \end{aligned} \quad (4.32)$$

where

$$d\bar{y}_2^2 = d\rho_1^2 + \rho_1^2 d\phi_1^2 , \quad (4.33)$$

and

$$\begin{aligned} A_r &= \mathcal{H}^{1/2} r^2 = \frac{\mathcal{G}\mathcal{H}^{1/2}}{J^2} \sin^2 \left( \frac{J}{\mathcal{G}^{1/2}\mathcal{H}^{1/2}} u \right) , \\ A_x &= \mathcal{H}^{1/2} (\mathcal{G} - J^2 r^2) = \mathcal{G}\mathcal{H}^{1/2} \cos^2 \left( \frac{J}{\mathcal{G}^{1/2}\mathcal{H}^{1/2}} u \right) , \\ A_y &= \mathcal{H}^{1/2} , \quad B_y = 4|\hat{\beta}|^2 J , \\ A_\rho &= \mathcal{H}^{1/2} , \quad \tilde{A}_\rho = \mathcal{G}\mathcal{H}^{1/2} , \quad B_\rho = 4J , \\ C &= \frac{J^2[(1 - |\hat{\beta}|^2 - 4|\hat{\beta}|^4)\bar{y}_2^2 - 4|\hat{\beta}|^2\rho^2]}{\mathcal{H}^{1/2}} . \end{aligned} \quad (4.34)$$

In Brinkmann-like coordinates, the metric is given by

$$\begin{aligned} ds^2 = & 2dudv + d\bar{r}_4^2 + d\bar{y}_2^2 + d\rho^2 + d\tilde{\rho}^2 + G_y (y_1 dy_2 - y_2 dy_1) du + G_\rho \rho d\tilde{\rho} du \\ & + (F_r \bar{r}_4^2 + F_y \bar{y}_2^2 + F_\rho \rho^2) du^2 , \end{aligned} \quad (4.35)$$

where  $\vec{r}_4 = (\bar{r}_3, x)$  and

$$\begin{aligned} F_r &= -\frac{J^2}{\mathcal{G}\mathcal{H}} , \\ F_y &= -\frac{J^2(1 - |\hat{\beta}|^2 - 4|\hat{\beta}|^4)}{\mathcal{H}} , \quad G_y = \frac{4J|\hat{\beta}|^2}{\mathcal{H}^{1/2}} , \\ F_\rho &= \frac{4J^2|\hat{\beta}|^2}{\mathcal{H}} , \quad G_\rho = \frac{4J}{\mathcal{G}^{1/2}\mathcal{H}^{1/2}} , \end{aligned} \quad (4.36)$$

while the remaining fields read

$$\begin{aligned} e^{2\Phi} &= \mathcal{G}\mathcal{H}^2 g_s^2 \\ F_5 &= \frac{J}{\mathcal{G}^{1/2}\mathcal{H}^{3/2} g_s} du \wedge (dr_1 \wedge dr_2 \wedge dr_3 \wedge dr_4 - dy_1 \wedge dy_2 \wedge d\rho \wedge d\tilde{\rho}) . \end{aligned} \quad (4.37)$$

We see that only the dilaton and the RR 5-form field strength survive the Penrose limit along this particular geodesic. The deformation affects all the  $F$ -functions in the metric and all nonzero fields.

## Two equal rotation parameters (sphere)

We next consider the case of two equal rotation parameters, where the new parameter entering the problem is  $r_0$ . The results for the various geodesics are as follows:

- **$(J, 0, 0)$  geodesic.** The differential equation for  $r$  becomes

$$\dot{r}^2 = \Delta_-^2 - J^2 r^2 , \quad \Delta_- \equiv \sqrt{1 - \frac{r_0^2}{r^2}} . \quad (4.38)$$

Its solution is then given by

$$r^2(u) = \frac{1}{2J^2}(1 - a \cos 2Ju) , \quad a \equiv \sqrt{1 - 4J^2 r_0^2} , \quad (4.39)$$

from which it follows that

$$\Delta_-(u) = \sqrt{1 + \frac{a^2 - 1}{2(1 - a \cos 2Ju)}} . \quad (4.40)$$

Note that reality requires that, for fixed  $r_0$ , there is a maximum angular momentum associated with this trajectory. In Rosen coordinates, the Penrose limit of the deformed metric reads

$$ds^2 = 2dudv + A_r d\bar{r}_3^2 + A_x dx^2 + d\bar{y}_4^2 - C du^2 , \quad (4.41)$$

where

$$d\bar{y}_4^2 = d\rho^2 + \rho^2(d\psi^2 + \sin^2 \psi d\phi_2^2 + \cos^2 \psi d\phi_3^2) , \quad (4.42)$$

and

$$A_r = r^2 , \quad A_x = \Delta_-^2 - J^2 r^2 , \quad C = (1 + 4|\hat{\beta}|^2)J^2 \bar{y}_4^2 , \quad (4.43)$$

and are to be understood as functions of  $u$  through the identifications (4.39) and (4.40).

In Brinkmann coordinates, the metric is given by

$$ds^2 = 2dudv + d\bar{r}_3^2 + dx^2 + d\bar{y}_4^2 + (F_r \bar{r}_3^2 + F_x x^2 + F_y \bar{y}_4^2)du^2 , \quad (4.44)$$

with

$$\begin{aligned} F_r &= -J^2 \left[ 1 + \frac{a^2 - 1}{(1 - a \cos 2Ju)^2} \right] , \\ F_x &= -J^2 \left[ 1 - 3 \frac{a^2 - 1}{(1 - a \cos 2Ju)^2} \right] , \\ F_y &= -(1 + 4|\hat{\beta}|^2)J^2 . \end{aligned} \quad (4.45)$$

Note that, since  $0 < a < 1$  the metric is no-where singular. The remaining nonzero fields are

$$\begin{aligned}
H_3 &= -4J\hat{\gamma}du \wedge (dy_1 \wedge dy_2 - dy_3 \wedge dy_4) , \\
e^{2\Phi} &= g_s^2 , \\
F_3 &= \frac{4J\hat{\sigma}}{g_s} du \wedge (dy_1 \wedge dy_2 - dy_3 \wedge dy_4) , \\
F_5 &= \frac{J}{g_s} du \wedge (dr_1 \wedge dr_2 \wedge dr_3 \wedge dx - dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4) .
\end{aligned} \tag{4.46}$$

These are the same as in Eq. (4.18) for the same geodesic at zero rotation. This is in agreement with the results of [27], corresponding to the limiting case  $\gamma = \sigma = 0$ , where it was noted that the RR 5-form field strength at the Penrose limit for the  $(J, 0, 0)$  geodesic retains the same form as in the zero-rotation case. Indeed, since the Penrose limit and the  $\beta$ -deformation procedure commute, the fact that in the undeformed case the Penrose limits of the metric along the torus directions, of the dilaton and of the RR 5-form are the same as at zero rotation guarantees that, in the deformed case, the fields in (4.46) will be equal to those in (4.18). The deformation affects only the function  $F_y$  in the metric and the NSNS and RR 3-form field strengths while the effect of nonzero rotation parameters manifests itself only in the functions  $F_r$  and  $F_x$  in the metric.

- **$(0, 0, J)$  geodesic.** The differential equation for  $r$  becomes

$$\dot{r}^2 = 1 - J^2 r^2 \Delta_-^2 , \tag{4.47}$$

with  $\Delta_-$  as in (4.38). Its solution is

$$r^2(u) = \frac{b^2}{J^2} \sin^2 Ju , \quad b \equiv \sqrt{1 + J^2 r_0^2} , \tag{4.48}$$

from which it follows that

$$\Delta_-(u) = \sqrt{1 - \frac{b^2 - 1}{b^2 \sin^2 Ju}} . \tag{4.49}$$

The Penrose limit of the deformed metric reads

$$ds^2 = 2dudv + A_r d\vec{r}_3^2 + A_x dx^2 + A_y d\vec{y}_2^2 + \tilde{A}_y d\vec{y}_2^2 - C du^2 , \tag{4.50}$$

where

$$d\vec{y}_2^2 = dy_1^2 + dy_2^2 = d\rho_1^2 + \rho_1^2 d\phi_1^2 , \quad d\vec{y}_2^2 = dy_3^2 + dy_4^2 = d\rho_2^2 + \rho_2^2 d\phi_2^2 , \tag{4.51}$$

and

$$A_r = r^2 \Delta_- , \quad A_x = \frac{1}{\Delta_-} - J^2 r^2 \Delta_- , \quad A_y = \Delta_- , \quad \tilde{A}_y = \frac{1}{\Delta_-} ,$$

$$C = J^2 \left[ \left( \frac{\vec{y}_2^2}{\Delta_-} + \vec{\vec{y}}_2^2 \Delta_- \right) + 4|\hat{\beta}|^2 \left( \vec{y}_2^2 \Delta_- + \frac{\vec{\vec{y}}_2}{\Delta_-} \right) \right] ,$$

and again are to be understood as functions of  $u$  through (4.47) and (4.48). In Brinkmann coordinates, the metric reads

$$ds^2 = 2dudv + d\vec{r}_3^2 + dx^2 + d\vec{y}_2^2 + d\vec{\vec{y}}_2^2 + (F_r \vec{r}_3^2 + F_x x^2 + F_y \vec{y}_2^2 + \tilde{F}_y \vec{\vec{y}}_2^2) du^2 , \quad (4.52)$$

with

$$\begin{aligned} F_r &= -J^2 \left[ 1 + \frac{b^2 - 1}{4(1 - b^2 \cos^2 Ju)^2} \left( \frac{b^2 - 1}{\sin^2 Ju} - b^2 + 3 \right) \right] , \\ F_x &= -J^2 \left[ 1 - \frac{5}{4} \frac{b^2 - 1}{(1 - b^2 \cos^2 Ju)^2} - \frac{1}{4} \frac{b^2 - 1}{\sin^2 Ju (1 - b^2 \cos^2 Ju)} \right] , \\ F_y &= -J^2 \left[ \frac{b^2 \sin^2 Ju}{1 - b^2 \cos^2 Ju} - \frac{(b^2 - 1)(4b^2 \cos^4 Ju - 2 - (b^2 + 1) \cos^2 Ju)}{4(1 - b^2 \cos^2 Ju)^2 \sin^2 Ju} + 4|\hat{\beta}|^2 \right] , \\ \tilde{F}_y &= -J^2 \left[ \frac{1 - b^2 \cos^2 Ju}{b^2 \sin^2 Ju} + \frac{(b^2 - 1)(4b^2 \cos^4 Ju - 2 + (1 - 3b^2) \cos^2 Ju)}{4(1 - b^2 \cos^2 Ju)^2 \sin^2 Ju} + 4|\hat{\beta}|^2 \right] , \end{aligned} \quad (4.53)$$

while the remaining nonzero fields are

$$\begin{aligned} H_3 &= -4J\hat{\gamma} du \wedge (dy_1 \wedge dy_2 - dy_3 \wedge dy_4) , \\ e^{2\Phi} &= g_s^2 , \\ F_3 &= \frac{4J\hat{\sigma}}{g_s} du \wedge (dy_1 \wedge dy_2 - dy_3 \wedge dy_4) , \\ F_5 &= \frac{J}{2g_s} \frac{1 - b^2 \cos 2Ju}{b \sin Ju \sqrt{1 - b^2 \cos^2 Ju}} du \wedge (dr_1 \wedge dr_2 \wedge dr_3 \wedge dx - dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4) . \end{aligned} \quad (4.54)$$

In contrast to the previous case, these differ from the corresponding expression for the same geodesic at zero rotation in that the RR 5-form now has an  $r_0$ -dependent overall coefficient. Again this is in agreement with the corresponding analysis of [27] for the limiting case  $\gamma = \sigma = 0$ . Now, the deformation affects only the functions  $F_y$  and  $\tilde{F}_y$  in the metric and the NSNS and RR 3-form field strengths while the effect of turning on rotation parameters manifests itself in all  $F$ -functions in the metric and in the RR 5-form field strength.

• **(0,  $J$ ,  $J$ ) geodesic.** Now, it is convenient to introduce

$$\mathcal{G}^{-1} = \frac{(1 + |\hat{\beta}|^2)r^2 - r_0^2}{r^2 - r_0^2} , \quad \mathcal{H} = \frac{(1 + \hat{\sigma}^2)r^2 - r_0^2}{r^2 - r_0^2} , \quad (4.55)$$

which are the effective values of the functions in (3.27) for the given ansatz, now being functions of  $r$ . The differential equation for  $r$  has the form

$$\mathcal{H}\dot{r}^2 = 1 - \frac{J^2 r^2 \Delta_-^2}{\mathcal{G}} , \quad (4.56)$$

with  $\Delta_-$  as in (4.38). Unfortunately, this equation cannot be solved explicitly in the general case, for which we will content ourselves with presenting the Penrose limit only in the Rosen-like form, with the  $u$ -dependence of the metric components entering implicitly through (4.54). This metric is given by

$$\begin{aligned} ds^2 = & 2dudv + A_r d\bar{r}_3^2 + A_x dx^2 + A_y d\bar{y}_2^2 + B_y (y_1 dy_2 - y_2 dy_1) du \\ & + A_\rho d\rho^2 + \tilde{A}_\rho d\tilde{\rho}^2 + B_\rho \rho d\tilde{\rho} du - C du^2 , \end{aligned} \quad (4.57)$$

where

$$d\bar{y}_2^2 = d\rho_1^2 + \rho_1^2 d\phi_1^2 , \quad (4.58)$$

and

$$\begin{aligned} A_r &= \mathcal{H}^{1/2} r^2 \Delta_- , \quad A_x = \mathcal{H}^{1/2} \left( \frac{\mathcal{G}}{\Delta_-} - J^2 r^2 \Delta_- \right) , \\ A_y &= \mathcal{H}^{1/2} \Delta_- , \quad B_2 = 4|\hat{\beta}|^2 J , \\ A_\rho &= \frac{\mathcal{H}^{1/2}}{\Delta_-} , \quad \tilde{A}_w = \frac{\mathcal{G}\mathcal{H}^{1/2}}{\Delta_-} , \quad B_\rho = 4J , \\ C &= \frac{J^2[(1 - |\hat{\beta}|^2 - 4|\hat{\beta}|^4)\bar{y}_2^2 - 4|\hat{\beta}|^2 \rho^2]}{\mathcal{H}^{1/2} \Delta_-} . \end{aligned} \quad (4.59)$$

On the other hand, for the case of a pure  $\gamma$ -deformation, Eq. (4.56) *can* be solved exactly with the result

$$r^2(u) = \frac{b^2 \mathcal{G}_0}{J^2} \sin^2 \left( \frac{Ju}{\mathcal{G}_0^{1/2}} \right) , \quad (4.60)$$

where  $b$  is as in (4.48) and

$$\mathcal{G}_0^{-1} = 1 + \hat{\gamma}^2 . \quad (4.61)$$

The various functions for the metric in Rosen-like coordinates are found by substituting (4.60) into (4.57) and setting  $\mathcal{G} \rightarrow \mathcal{G}_0$  and  $\mathcal{H} \rightarrow 1$ . In Brinkmann-like coordinates, the formulas for the various functions are rather messy and we refrain from quoting them. We just note that the solution has  $F_1 = F_3 = 0$  and that all  $F$ -functions in the metric and all nonzero fields are affected both by the deformation and by the rotation parameters.

## One rotation parameter (disc)

We finally consider the case of one rotation parameter. As remarked earlier on, when  $\mu = 0$ , the solutions for one rotation parameter are obtained from those for two equal rotation parameters through the replacement  $r_0^2 \rightarrow -r_0^2$ . Making this replacement in the corresponding Penrose limits, we find the following results:

- **$(J, 0, 0)$  geodesic.** The Penrose limit of the solution in Brinkmann coordinates is given by Eqs. (4.44)–(4.46), now with

$$a = \sqrt{1 + 4J^2r_0^2}. \quad (4.62)$$

Note that now  $a$  is real for all values of  $Jr_0$ . Also, since  $a > 1$  the PP-wave metric is singular at  $\cos 2Ju = 1/a$ .

- **$(0, 0, J)$  geodesic.** The Penrose limit of the solution in Brinkmann coordinates is given by Eqs. (4.52)–(4.54), now with

$$b = \sqrt{1 - J^2r_0^2}. \quad (4.63)$$

- **$(0, J, J)$  geodesic.** In the general case, the Penrose limit of the deformed metric in Rosen-like coordinates is given by Eq. (4.57) with  $d\bar{y}_2^2$  given in (4.58) and the  $A_-$ ,  $B_-$  and  $C_-$  functions given in (4.59) but with  $\Delta_-$  replaced by  $\Delta_+ = \sqrt{1 + \frac{r_0^2}{r^2}}$  and with  $\mathcal{G}^{-1}$  and  $\mathcal{H}$  appropriately modified. For the case of a pure  $\gamma$ -deformation, there exists the explicit solution (4.58) for  $r(u)$  and the Penrose limit in Rosen-like coordinates is found by substituting that solution into Eqs. (4.57)–(4.59), with  $\Delta_-$  replaced by  $\Delta_+$ .

## 5 Giant gravitons on $\beta$ -deformed PP-waves

Given the marginally-deformed geometries, it is interesting to investigate the various extended objects that they can support. In this respect, an important role is played by BPS configurations of spherical D3-branes, the so-called giant gravitons. To summarize the basic facts, the authors of [29], building on results of [30], considered a KK excitation (graviton) in  $\text{AdS}_5 \times S^5$  with nonzero angular momentum along an  $S^5$  direction and contemplated the possibility that it might blow up on an  $S^3$  inside the  $S^5$  without raising its energy. They found that such a state (the giant graviton) can indeed exist, with the blowing up being due to its angular momentum and the extra force required to keep it stable under shrinking being provided by RR repulsion. Soon after that, it was found

[31] that there also exist “dual” giant gravitons with similar properties, supported on the  $S^3$  inside the  $AdS_5$  part of the geometry. The construction of giant gravitons has been extended towards various directions in [32], while investigations from the dual field-theory side have been carried out in [33]. An interesting result is that, in certain cases [34, 35] where the geometry of the  $S^5$  supporting the giant graviton is deformed, the latter has higher energy than the point graviton, and may even not exist at all as a solution.

This latter fact serves as a motivation for examining giant gravitons in the marginally-deformed solutions of interest, since one of the main features of the latter is precisely a deformation of  $S^5$ . In the existing literature, giant gravitons have been considered only for the case of  $\gamma$ -deformations. For the full deformed solutions, giant gravitons were first constructed in [35] for a special case of the non-supersymmetric three-parameter background of [4, 5] and, more recently, for the general three-parameter [36] and the single parameter [36, 37] backgrounds; in particular, the giant gravitons of [36, 37] were found to be independent of the deformation parameter and hence still degenerate with the point graviton. For the PP-wave limits of the marginally-deformed backgrounds, giant gravitons have been constructed, in analogy to the considerations of [38] for the PP-wave limits of  $AdS_5 \times S^5$ , in [39] for the two PP-wave limits of  $\gamma$ -deformed  $AdS_5 \times S^5$ , namely those along the  $(J, 0, 0)$  and the  $(J, J, J)$  geodesic. For the first geodesic, the solution was constructed exactly and it was found that  $\gamma$ -deformations do not lift the degeneracy of the giant and point gravitons, in accordance with [37], and a stability analysis indicated that the former is perturbatively stable. For the second geodesic, the problem was attacked perturbatively in  $\hat{\gamma}$ , with the leading-order analysis indicating that the  $\gamma$ -deformation lifts the degeneracy in favor of the point graviton, but no definitive conclusion on whether the giant graviton survives the deformation for large enough  $\hat{\gamma}$  was reached.

Here, we address the question of identifying giant gravitons on the PP-wave limit of the deformed geometries, this time in the presence of  $\sigma$ -deformations and/or nonzero rotation parameters. Restricting to PP-waves along the  $(J, 0, 0)$  geodesic, our exact analysis for the giant graviton residing on the deformed  $S^5$  part of the geometry shows that  $\sigma$ -deformations have an altogether different effect than that of  $\gamma$ -deformations, lifting the degeneracy of the giant and point gravitons and, for  $\hat{\sigma}$  above a critical value, completely removing the giant graviton from the spectrum. Moreover, the analysis of small fluctuations of the giant graviton reveals that the latter is perturbatively stable throughout its range of existence. We also consider dual giant gravitons residing on the

AdS<sub>5</sub> part of the geometry, in which case the deformation does not affect neither the solution, nor its stability properties. In what follows, we present the relevant analysis, keeping a similar notation with [39] in order to facilitate comparison.

## 5.1 Giant gravitons on the deformed PP–waves

To describe the giant-graviton solutions in the PP–wave spacetimes of interest, we consider the action for a probe D3-brane in this background, given by the sum of the Dirac–Born–Infeld and Wess–Zumino terms,

$$S_{\text{D3}} = S_{\text{DBI}} + S_{\text{WZ}} = -T_3 \int d^4\sigma e^{-\Phi} \sqrt{-\det \mathcal{P}[G - B]} + T_3 \int \sum_q \mathcal{P}[A_q \wedge e^{-B_2}] , \quad (5.1)$$

where  $T_3$  is the D3-brane tension, equal to  $1/(2\pi)^3$  in units where  $\alpha' = 1$ , and  $\mathcal{P}[f]$  stands for the pullback of a spacetime field  $f$  on the worldvolume. Below we describe the construction of giant gravitons and dual giant gravitons on the deformed PP–waves under consideration.

### 5.1.1 Giant gravitons

Starting with ordinary giant gravitons, we want to describe a D3-brane wrapping the S<sup>3</sup> inside the S <sub>$\beta$</sub> <sup>5</sup> in the deformed geometry. To do so, we employ the gauge choice

$$\tau = u , \quad \sigma_1 = \psi , \quad \sigma_2 = \phi_2 , \quad \sigma_3 = \phi_3 , \quad (5.2)$$

and we consider the ansatz

$$v = -\nu u , \quad \vec{r}_4 = 0 , \quad \rho = \rho_0 = \text{const.} . \quad (5.3)$$

Noting that the spatial brane coordinates are just the angular coordinates employed in the Rosen form (4.12)–(4.15) of the PP–wave solution of interest, the pullbacks of the various fields on the D3-brane can be immediately read off from these equations. Setting for convenience  $J = 1$ , we find

$$\mathcal{P}[G] = \text{diag} \left( -2\nu - (1 + 4|\hat{\beta}|^2)\rho_0^2, \rho_0^2, \rho_0^2 \sin^2 \psi, \rho_0^2 \cos^2 \psi \right) , \quad (5.4)$$

and

$$\begin{aligned} \mathcal{P}[B_2] &= 2\hat{\gamma}\rho_0^2 du \wedge (\sin^2 \psi d\sigma_2 - \cos^2 \psi d\sigma_3) , \\ \mathcal{P}[A_2] &= -\frac{2\hat{\sigma}}{g_s} \rho_0^2 du \wedge (\sin^2 \psi d\sigma_2 - \cos^2 \psi d\sigma_3) , \\ \mathcal{P}[A_4] &= \frac{1}{g_s} \rho_0^4 \cos \psi \sin \psi du \wedge d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3 . \end{aligned} \quad (5.5)$$

From these equations, we readily compute

$$-\det \mathcal{P}[G - B] = [2\nu + (1 + 4\hat{\sigma}^2)\rho_0^2]\rho_0^6 \cos^2 \psi \sin^2 \psi . \quad (5.6)$$

As in [39], we note that all  $\hat{\gamma}$  dependence has dropped out due to the cancellation of terms coming from the metric and from  $B_2$ . The important fact is that, since the metric now involves  $|\hat{\beta}|^2$  in place of  $\hat{\gamma}^2$  while  $B_2$  still depends only on  $\hat{\gamma}$ , there remains a non-trivial dependence on the deformation through the parameter  $\hat{\sigma}$ . Noting also that, since  $\mathcal{P}[B_2 \wedge A_2] = 0$ , only  $\mathcal{P}[A_4]$  contributes to the Wess-Zumino action, we write the full D3-brane action as

$$S_{\text{D3}} = -\frac{T_3}{g_s} \int du d\Omega_3 [\rho_0^3 \sqrt{2\nu + (1 + 4\hat{\sigma}^2)\rho_0^2} - \rho_0^4] = \int du L_{\text{D3}} , \quad (5.7)$$

where  $d\Omega_3 = \cos \psi \sin \psi d\psi d\phi_1 d\phi_2$  is the  $S^3$  volume element and  $L_{\text{D3}}$  is the Lagrangian

$$L_{\text{D3}} = -M[\rho_0^3 \sqrt{2\nu + (1 + 4\hat{\sigma}^2)\rho_0^2} - \rho_0^4] , \quad (5.8)$$

where

$$M = \frac{2\pi^2 T_3}{g_s} = \frac{1}{4\pi g_s} = \frac{N}{R^4} . \quad (5.9)$$

Since  $L_{\text{D3}}$  is independent of  $v$  and  $u$ , we have two conserved first integrals, given by the light-cone momentum

$$P = -\frac{\partial L_{\text{D3}}}{\partial \nu} = \frac{M\rho_0^3}{\sqrt{2\nu + (1 + 4\hat{\sigma}^2)\rho_0^2}} , \quad (5.10)$$

and the light-cone Hamiltonian

$$E = \nu \frac{\partial L_{\text{D3}}}{\partial \nu} - L_{\text{D3}} = \frac{M\rho_0^3 [\nu + (1 + 4\hat{\sigma}^2)\rho_0^2]}{\sqrt{2\nu + (1 + 4\hat{\sigma}^2)\rho_0^2}} - M\rho_0^4 . \quad (5.11)$$

Solving (5.10) for  $\nu$  and substituting in (5.11), we write

$$E = \frac{M^2}{2P}\rho_0^6 - M\rho_0^4 + \frac{(1 + 4\hat{\sigma}^2)P}{2}\rho_0^2 . \quad (5.12)$$

As a function of  $\rho_0$ ,  $E$  has local extrema at the radii

$$\rho_0 = 0, \quad \rho_0 = \rho_{0\pm} = \sqrt{\frac{(2 \pm \Delta)P}{3M}} , \quad (5.13)$$

where we have defined

$$\Delta \equiv \sqrt{1 - 12\hat{\sigma}^2} . \quad (5.14)$$

The corresponding light-cone energies are given by

$$E_0 = 0 , \quad E_{\pm} = (2 \pm \Delta)^2 (1 \mp \Delta) \frac{P^2}{27M} . \quad (5.15)$$

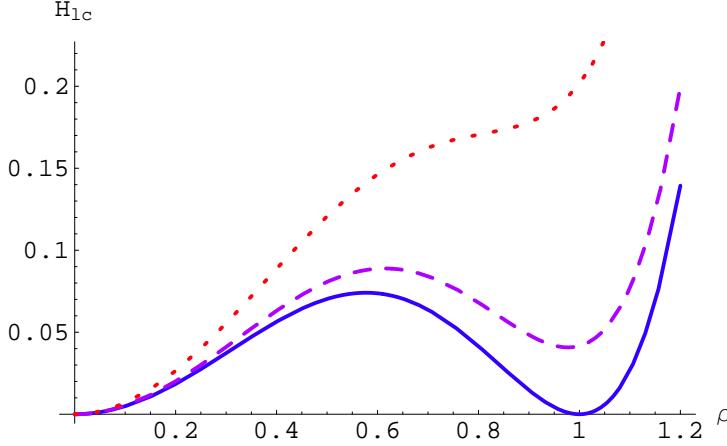


Figure 1: Light-cone Hamiltonian for the PP-wave along the  $(J, 0, 0)$  geodesic plotted as a function of  $\rho$ . The three curves correspond to  $\hat{\sigma} = 0$  (solid),  $\hat{\sigma} = \frac{0.5}{2\sqrt{3}}$  (dashed) and  $\hat{\sigma} = \frac{1.1}{2\sqrt{3}}$  (dotted). The plots are shown in units where  $M = P = 1$

Note also that (5.10) and the second of (5.13) lead to the constraint

$$\nu = -\frac{2}{9}\rho_{0\pm}^2 (2 \pm \Delta) (1 \mp \Delta) , \quad (5.16)$$

determining  $\nu$  in terms of  $\rho_{0\pm}$ .

For  $0 \leq \hat{\sigma} < \frac{1}{2\sqrt{3}}$ , the radius  $\rho_0 = \rho_{0-}$  corresponds to a local maximum, while the radii  $\rho_0 = 0$  and  $\rho_0 = \rho_{0+}$  correspond to two local minima, the point graviton and the giant graviton. At exactly  $\hat{\sigma} = 0$ , we recover the usual result that  $E_+ = E_0$ , i.e. that the giant graviton is degenerate in energy with the point graviton. However, for  $0 < \hat{\sigma} < \frac{1}{2\sqrt{3}}$ , we have  $E_+ > E_0$  i.e. the degeneracy is lifted with the giant graviton becoming energetically unfavorable. At  $\hat{\sigma} = \frac{1}{2\sqrt{3}}$ , the radii  $\rho_0 = \rho_{0-}$  and  $\rho_0 = \rho_{0+}$  degenerate into a saddle point, leaving only one minimum at  $\rho_0 = 0$ . Finally, for  $\hat{\sigma} > \frac{1}{2\sqrt{3}}$ , the only extremum is the minimum at  $\rho_0 = 0$ : the giant graviton disappears from the spectrum. The situation is depicted in Fig. 1. To summarize, for the PP-wave along the  $(J, 0, 0)$  geodesic, complex  $\beta$ -deformations, unlike  $\gamma$ -deformations, have the effect of lifting the degeneracy of the giant and point gravitons for small values of  $\hat{\sigma}$  and of removing the giant graviton from the spectrum for large values of  $\hat{\sigma}$ . Also, as we shall see explicitly later on, although the effective Lagrangian (5.8) depends only on  $\hat{\sigma}$ , the spectrum of small perturbations about the giant graviton solution is dependent on both deformation parameters.

We note that the above results remain valid when the rotation parameter  $r_0$  is turned on. This follows from the fact that the relevant components of the metric and the remaining fields are identical to those at zero rotation (see the comments following Eq. (4.46)).

### 5.1.2 Dual giant gravitons

Proceeding to the case of dual giant gravitons, we have to consider a D3-brane wrapping the  $S^3$  originating from the  $AdS_5$  part of the geometry. Since the latter part of the geometry is unaffected by the deformation, it is immediately seen that the dual giant graviton solution exists and is independent of the deformation; however, to set up the notation for the stability analysis that follows, let us demonstrate it explicitly. To do so, we first parametrize, in analogy to (4.13), the coordinate vector  $\vec{r}_4$  by the coordinates  $(\tilde{\rho}, \tilde{\psi}, \tilde{\phi}_2, \tilde{\phi}_3)$  defined by

$$d\vec{r}_4^2 = d\tilde{\rho}^2 + \tilde{\rho}^2(d\tilde{\psi}^2 + \sin^2 \tilde{\psi} d\tilde{\phi}_2^2 + \cos^2 \tilde{\psi} d\tilde{\phi}_3^2) , \quad (5.17)$$

we employ the gauge choice

$$\tau = u , \quad \sigma_1 = \tilde{\psi} , \quad \sigma_2 = \tilde{\phi}_2 , \quad \sigma_3 = \tilde{\phi}_3 , \quad (5.18)$$

and we consider the ansatz

$$v = -\nu u , \quad \vec{y}_4 = 0 , \quad \tilde{\rho} = \tilde{\rho}_0 = \text{const.} . \quad (5.19)$$

Proceeding as before, we find that the only relevant pullbacks of the spacetime fields are

$$\mathcal{P}[G] = \text{diag} \left( -2\nu - \tilde{\rho}_0^2, \tilde{\rho}_0^2, \tilde{\rho}_0^2 \sin^2 \tilde{\psi}, \tilde{\rho}_0^2 \cos^2 \tilde{\psi} \right) , \quad (5.20)$$

and

$$\mathcal{P}[A_4] = \frac{1}{g_s} \tilde{\rho}_0^4 \cos \tilde{\psi} \sin \tilde{\psi} du \wedge d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3 , \quad (5.21)$$

and we arrive at the action

$$S_{D3} = \int du L_{D3} , \quad L_{D3} = -M \left( \tilde{\rho}_0^3 \sqrt{2\nu + \tilde{\rho}_0^2} - \tilde{\rho}_0^4 \right) , \quad (5.22)$$

with  $M$  as in (5.9). The dual giant graviton solution is found as before and is obviously independent of the deformation, which implies that it is degenerate with the point graviton for all values of the deformation parameters. However, as we shall see, this degeneracy does not extend to the spectrum of fluctuations around this solution.

When the rotation parameter  $r_0$  is turned on, however, the above solution is no longer valid. This is because the  $SO(4)$  symmetry of the  $\vec{r}_4 = (\vec{r}_3, x)$  directions is broken by the rotation parameter, as manifested by the fact that the functions  $F_r(u)$  and  $F_x(u)$  in (4.44) are different.

## 5.2 Small fluctuations and perturbative stability

We now turn to an analysis of small (bosonic) fluctuations about the giant graviton configurations in deformed PP-waves, following the treatment of [40]. We consider both ordinary and dual giant gravitons and, for the former case, we also take account of the presence of rotation. As we shall see, although the spectrum of fluctuations is affected by the deformation, the standard result that these configurations are perturbatively stable remains unchanged.

### 5.2.1 Giant gravitons

Starting with ordinary giant gravitons, the perturbation of the classical configuration is described by keeping the gauge choice as in (5.2) and perturbing the embedding as

$$\begin{aligned} v &= -\nu u + \delta v(u, \psi, \phi_2, \phi_3) , & \rho &= \rho_0 + \delta \rho(u, \psi, \phi_2, \phi_3) , \\ \vec{r}_3 &= \delta \vec{r}_3(u, \psi, \phi_2, \phi_3) , & x &= \delta x(u, \psi, \phi_2, \phi_3) . \end{aligned} \quad (5.23)$$

Computing the pullbacks of the various fields as before, inserting them in the D3-brane action and expanding up to second order in the fluctuations, we write

$$S_{\text{D3}} = S_0 + S_1 + S_2 + \dots , \quad (5.24)$$

where the various terms correspond to the respective powers of the fluctuations. The zeroth-order is just the classical action given by (5.7) and (5.8). The linear term reads

$$\begin{aligned} S_1 &= -\frac{M \rho_0^2}{2\pi^2 \sqrt{2\nu + (1 + 4\hat{\sigma}^2)\rho_0^2}} \\ &\times \int du d\Omega_3 \left\{ \rho_0 \partial_u \delta v - 2 \left[ 3\nu + 2(1 + 4\hat{\sigma}^2)\rho_0^2 - 2\rho_0 \sqrt{2\nu + (1 + 4\hat{\sigma}^2)\rho_0^2} \right] \delta \rho \right\} , \end{aligned} \quad (5.25)$$

and it depends only on the deformation parameter  $\hat{\sigma}$ . The first term is a total derivative that vanishes upon integration over  $u$ , while the second term vanishes upon imposing the constraint (5.16). Finally, calculating the quadratic term and making use of (5.16) (with

the upper signs), we obtain

$$\begin{aligned}
S_2 = & -\frac{3M\rho_0^2}{4\pi^2(2+\Delta)} \int dud\Omega_3 \left\{ \frac{8\Delta(1-\Delta)}{3} \delta\rho^2 - F_r(u) \delta\vec{r}_3^2 - F_x(u) \delta x^2 \right. \\
& - \frac{12\Delta}{(2+\Delta)\rho_0} \delta\rho \partial_u \delta v + 4\hat{\gamma}^2 [(\partial_{\phi_2} - \partial_{\phi_3}) \delta\vec{r}_5]^2 \\
& - \frac{9}{(2+\Delta)^2 \rho_0^2} (\partial_u \delta v)^2 - (\partial_u \delta\vec{r}_5)^2 \\
& \left. + \frac{1}{\rho_0^2} h^{\alpha\beta} \partial_\alpha \delta v \partial_\beta \delta v + \frac{(2+\Delta)^2}{9} h^{\alpha\beta} \partial_\alpha \delta\vec{r}_5 \cdot \partial_\beta \delta\vec{r}_5 \right\} . \tag{5.26}
\end{aligned}$$

Here,  $\Delta$  is the  $\hat{\sigma}$ -dependent quantity defined in (5.14),  $h_{\alpha\beta}$  is the metric on  $S^3$  and we introduced the shorthand  $\delta\vec{r}_5 = (\delta\vec{r}_3, \delta x, \delta\rho)$ . Also,  $F_r(u)$  and  $F_x(u)$  are the  $u$ -dependent functions defined in (4.45), with the parameter  $a$  given by the second of (4.39) and by (4.62) for the case of two and one rotation parameters respectively. In the limit of zero rotation both  $F_r$  and  $F_x$  equal to  $-1$ . We note that, unlike the zeroth-order and linear terms, the quadratic term has some, *albeit* restricted, dependence on the deformation parameter  $\hat{\gamma}$  in addition to the dependence on  $\hat{\sigma}$ . After several integrations by parts, the quadratic action becomes

$$\begin{aligned}
S_2 = & -\frac{3M\rho_0^2}{4\pi^2(2+\Delta)} \int dud\Omega_3 \left\{ \frac{8\Delta(1-\Delta)}{3} \delta\rho^2 - F_r(u) \delta\vec{r}_3^2 - F_x(u) \delta x^2 \right. \\
& - \frac{12\Delta}{(2+\Delta)\rho_0} \delta\rho \partial_u \delta v + \frac{1}{\rho_0^2} \delta v \left[ \left( \frac{3}{2+\Delta} \right)^2 \partial_u^2 - \Delta_{S^3} \right] \delta v \tag{5.27} \\
& \left. + \delta\vec{r}_5 \cdot \left[ \partial_u^2 - \left( \frac{2+\Delta}{3} \right)^2 \Delta_{S^3} - 4\hat{\gamma}^2 (\partial_{\phi_2} - \partial_{\phi_3})^2 \right] \delta\vec{r}_5 \right\} ,
\end{aligned}$$

where  $\Delta_{S^3}$  is the Laplacian on  $S^3$ . To proceed, we may expand all fluctuations in the basis spanned by the combinations  $\Psi_{S^3, \ell n_2 n_3}(\psi, \phi_2, \phi_3)$  of  $S^3$  harmonics having definite quantum numbers under the two  $U(1)$ 's corresponding to shifts of  $\phi_2$  and  $\phi_3$ , i.e. by the simultaneous eigenfunctions of the operators  $\Delta_{S^3}$  and  $\partial_{\phi_{2,3}}$  with

$$\begin{aligned}
\Delta_{S^3} \Psi_{S^3, \ell n_2 n_3}(\psi, \phi_2, \phi_3) &= -\ell(\ell+2) \Psi_{S^3, \ell n_2 n_3}(\psi, \phi_2, \phi_3) , \\
\partial_{\phi_{2,3}} \Psi_{S^3, \ell n_2 n_3}(\psi, \phi_2, \phi_3) &= i n_{2,3} \Psi_{S^3, \ell n_2 n_3}(\psi, \phi_2, \phi_3) , \tag{5.28}
\end{aligned}$$

where  $\ell$ ,  $n_2$  and  $n_3$  are required to satisfy<sup>4</sup>

$$\ell = 0, 1, \dots , \quad \ell - |n_2| - |n_3| = 2k , \quad k = 0, 1, \dots . \tag{5.29}$$

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<sup>4</sup>The explicit expression for the  $\Psi_{S^3, \ell n_2 n_3}$ 's can be written in terms of Jacobi Polynomials (see, for instance, section 6 of the first of [14]).

For the  $\delta v$  and  $\delta \rho$  fluctuations, the fact that their coefficients in the action are  $u$ -independent allows us to introduce an  $e^{-i\omega u}$  time dependence and thus we can write

$$\begin{aligned}\delta v(u, \psi, \phi_2, \phi_3) &= \delta v_{\ell n_2 n_3} e^{-i\omega u} \Psi_{S^3, \ell n_2 n_3}(\psi, \phi_2, \phi_3) , \\ \delta \rho(u, \psi, \phi_2, \phi_3) &= \delta \rho_{\ell n_2 n_3} e^{-i\omega u} \Psi_{S^3, \ell n_2 n_3}(\psi, \phi_2, \phi_3) ,\end{aligned}\quad (5.30)$$

while for the  $(\delta \vec{r}_3, \delta x)$  fluctuations, we can only set

$$\begin{aligned}\delta \vec{r}_3(u, \psi, \phi_2, \phi_3) &= \delta \vec{r}_{3, \ell n_2 n_3}(u) \Psi_{S^3, \ell n_2 n_3}(\psi, \phi_2, \phi_3) , \\ \delta x(u, \psi, \phi_2, \phi_3) &= \delta x_{\ell n_2 n_3}(u) \Psi_{S^3, \ell n_2 n_3}(\psi, \phi_2, \phi_3) .\end{aligned}\quad (5.31)$$

The spectrum of fluctuations follows from inserting the expansions (5.30) and (5.31) in the equations of motion stemming from the action (5.27). Starting from the null fluctuation  $\delta v$  and the radial fluctuation  $\delta \rho$ , we find that they satisfy the coupled system

$$\begin{pmatrix} \frac{(2+\Delta)^2 \ell(\ell+2)-9\omega^2}{(2+\Delta)^2 \rho_0^2} & -\frac{6i\Delta\omega}{(2+\Delta)\rho_0} \\ \frac{6i\Delta\omega}{(2+\Delta)\rho_0} & \frac{(2+\Delta)^2 \ell(\ell+2)-9\omega^2+36\hat{\gamma}^2(n_2-n_3)^2+24\Delta(1-\Delta)}{9} \end{pmatrix} \begin{pmatrix} \delta v_{\ell n_2 n_3} \\ \delta \rho_{\ell n_2 n_3} \end{pmatrix} = 0 , \quad (5.32)$$

which leads to the following spectrum

$$\begin{aligned}\omega_{\pm, \ell n_2 n_3}^2 &= \left(\frac{2+\Delta}{3}\right)^2 \ell(\ell+2) + 2 \left[ \frac{\Delta(2+\Delta)}{3} + \hat{\gamma}^2(n_2-n_3)^2 \right] \\ &\pm 2\sqrt{\Delta^2 \left(\frac{2+\Delta}{3}\right)^2 \ell(\ell+2) + \left[ \frac{\Delta(2+\Delta)}{3} + \hat{\gamma}^2(n_2-n_3)^2 \right]^2} .\end{aligned}\quad (5.33)$$

Obviously the  $\omega_{+, \ell n_2 n_3}^2$  are positive-definite while, for  $\hat{\sigma} = 0$  ( $\Delta = 1$ ), the  $\omega_{-, \ell n_2 n_3}^2$  are positive-semidefinite, with a zero mode occurring for  $\ell = 0$ . Therefore, the only potential source of instabilities for these fluctuations is one of the  $\omega_{-, \ell n_2 n_3}^2$  becoming negative for some value of  $\hat{\sigma}$ . This would only be possible if the inequality

$$P(x) \equiv [\ell(\ell+2) - 24]x^2 + 4[\ell(\ell+2) + 6]x + 4[\ell(\ell+2) + 9\hat{\gamma}^2(n_2-n_3)^2] < 0 , \quad (5.34)$$

could be satisfied for some  $x$  with  $0 \leq x \leq 1$ . It is easily seen that this cannot happen for any values of  $\ell$ . Turning to the  $(\delta \vec{r}_3, \delta x)$  fluctuations, we find that they satisfy the Schrödinger equations

$$\left[ -\frac{d^2}{du^2} + F_r(u) - \left(\frac{2+\Delta}{3}\right)^2 \ell(\ell+2) - 4\hat{\gamma}^2(n_2-n_3)^2 \right] \delta \vec{r}_{3, \ell n_2 n_3}(u) = 0 , \quad (5.35)$$

and

$$\left[ -\frac{d^2}{du^2} + F_x(u) - \left(\frac{2+\Delta}{3}\right)^2 \ell(\ell+2) - 4\hat{\gamma}^2(n_2-n_3)^2 \right] \delta x_{\ell n_2 n_3}(u) = 0 . \quad (5.36)$$

To examine them, we may consider the following cases.

- Zero rotation ( $r_0 = 0$ ). In the absence of rotation parameters (in which case  $F_r = F_x = -1$ ), the  $\delta\vec{r}_3$  and  $\delta x$  fluctuations obey the same Schrödinger equation with a  $u$ -independent potential. Introducing an  $e^{-i\omega u}$  dependence, we easily obtain the spectrum

$$\omega_{r,\ell n_2 n_3}^2 = 1 + 4\hat{\gamma}^2(n_2 - n_3)^2 + \left(\frac{2 + \Delta}{3}\right)^2 \ell(\ell + 2), \quad (5.37)$$

whence we verify that the  $\omega_{r,\ell n_2 n_3}^2$  are manifestly positive-definite, signifying stability against small perturbations in these directions. The  $\omega_{r,\ell n_2 n_3}^2$  are increasing functions of  $\hat{\gamma}$  but decreasing functions of  $\hat{\sigma}$  with the last term ranging from  $\ell(\ell + 2)$  at  $\hat{\sigma} = 0$  to  $\frac{4}{9}\ell(\ell + 2)$  at  $\hat{\sigma} = \frac{1}{2\sqrt{3}}$ . Note also that no zero mode is possible.

- Non-zero rotation parameters ( $r_0 \neq 0$ ). In this case, the Schrödinger equations (5.35) and (5.36) have periodic potentials  $F_r$  and  $F_x$ , respectively, given by (4.45) (with the parameter  $a$  in (4.39)) and fixed eigenvalue depending on the deformation parameters  $\hat{\gamma}$  and  $\hat{\sigma}$  as well as on the quantum numbers  $\ell$  and  $n_{2,3}$ .<sup>5</sup> From the shape of the potentials and since the fixed eigenvalue is non-negative, we infer stability. However, since the potentials are periodic the spectrum is continuous with mass gaps, which tend to zero as the parameter  $a \rightarrow 0$ . Hence, for fixed quantum numbers  $\ell$  and  $n_{2,3}$ , there exist ranges of the deformation parameters  $\hat{\gamma}$  and  $\hat{\sigma}$  for which any other solution than the vanishing one is not allowed.

In conclusion, despite the fact that  $\sigma$ -deformations render the giant graviton states energetically unfavorable, the small-fluctuation analysis indicates that these objects are perturbatively stable in the range of  $\hat{\sigma}$  where they are allowed to exist in the first place. Presumably, the effect of  $\sigma$ -deformations renders the giant gravitons metastable rather than unstable. To further investigate this aspect, one may seek “bounce-like” instanton solutions connecting the giant and the point graviton and calculate the tunneling probability by standard WKB methods. In doing so, one must appropriately take into account possible fermionic zero modes resulting from the breaking of supersymmetries by the instanton, which tend to suppress the tunneling rate.

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<sup>5</sup>Exactly the same equations appeared in [27] in the light-cone quantization of strings moving in these PP-wave backgrounds (for  $\hat{\gamma} = \hat{\sigma} = 0$ ).

### 5.2.2 Dual giant gravitons

We next consider dual giant gravitons which, as we recall, are independent of the deformation. Now, the perturbation can be described by keeping the gauge choice as in (5.17) and perturbing the embedding according to

$$v = -\nu u + \delta v(u, \tilde{\psi}, \tilde{\phi}_2, \tilde{\phi}_3), \quad \tilde{\rho} = \tilde{\rho}_0 + \delta \tilde{\rho}(u, \tilde{\psi}, \tilde{\phi}_2, \tilde{\phi}_3), \quad \vec{y}_4 = \delta \vec{y}_4(u, \tilde{\psi}, \tilde{\phi}_2, \tilde{\phi}_3). \quad (5.38)$$

Repeating the same steps as before, we find that the quadratic term in the action of the fluctuations reads

$$\begin{aligned} S_2 = & -\frac{M\tilde{\rho}_0^2}{4\pi^2} \int du d\tilde{\Omega}_3 \left[ (1 + 4|\hat{\beta}|^2)\delta\vec{y}_4^2 - \frac{4}{\tilde{\rho}_0}\delta\tilde{\rho}\partial_u\delta v \right. \\ & \left. + \frac{1}{\tilde{\rho}_0^2}\delta v(\partial_u^2 - \Delta_{S^3})\delta v + \delta\vec{y}_5 \cdot (\partial_u^2 - \Delta_{S^3})\delta\vec{y}_5 \right], \end{aligned} \quad (5.39)$$

where we introduced the shorthand  $\delta\vec{y}_5 = (\delta\vec{y}_4, \delta\tilde{\rho})$ . We see that the deformation enters only through a modification of the mass term of  $\delta\vec{y}_4$ . Introducing an  $e^{-i\omega u}$  time dependence and expanding the fluctuations on  $S^3$  as before, we find the fluctuation spectrum

$$\omega_{\pm, \ell n_2 n_3}^2 = \ell(\ell + 2) + 2 \pm 2\sqrt{\ell(\ell + 2) + 1}, \quad (5.40)$$

and

$$\omega_{y, \ell n_2 n_3}^2 = 1 + 4|\hat{\beta}|^2 + \ell(\ell + 2), \quad (5.41)$$

for the  $(\delta v, \delta\tilde{\rho})$  and  $\delta\vec{y}_4$  fluctuations respectively. This spectrum is manifestly positive-semidefinite, with  $\omega_{-, \ell n_2 n_3}^2$  having the expected zero mode for  $\ell = 0$ . We conclude that the deformation does not affect the stability of dual giant gravitons, its sole effect being just a raise of the energy of the  $\delta\vec{y}_4$  fluctuations.

## 6 Probing the deformed geometry with Wilson loops

To further investigate the effects of  $\beta$ -deformations we now turn to another, completely different, direction, of more phenomenological nature. Namely, we consider the potential for a static heavy  $q\bar{q}$  pair in the dual gauge theory, extracted from the expectation value of a rectangular Wilson loop extending along the Euclidean time direction and one space direction. On the gravity side, the Wilson loop expectation value is calculated [15, 41] by minimizing the Nambu–Goto action for a fundamental string propagating into the dual supergravity background, whose endpoints are constrained to lie on the two sides of the

Wilson loop. Below, we first briefly review the procedure for calculating Wilson loops in general supergravity backgrounds of interest [42, 14] and then we apply it to the case of  $\sigma$ -deformations of the Coulomb branch.

## 6.1 General formalism

As stated above, the calculation of a Wilson loop in the gravity approach amounts to extremizing the Nambu–Goto action (taking into account the contribution of the NSNS 2-form if necessary) for a string propagating in the dual geometry whose endpoints trace the loop. To describe the propagation of the string, we first fix reparametrization invariance by taking  $(\tau, \sigma) = (t, x)$ . We next need to find a suitable ansatz that is sensitive to  $\beta$ -deformations. Specializing to a radial trajectory, it is easily seen that the only available choice for the embedding is<sup>6</sup>

$$r = u(x) , \quad \theta = 0 , \quad \psi = \frac{\pi}{4} , \quad \phi_1 = \text{const.} , \quad \phi_2 = \phi_3 = \text{const.} , \quad \text{rest} = \text{const.} . \quad (6.1)$$

We next pass to the Euclidean using the analytic continuation  $t \rightarrow it$ .<sup>7</sup> Then, the Nambu–Goto action is found to be

$$S = \frac{T}{2\pi} \int dx \sqrt{f(u)/R^4 + g(u)u'^2} , \quad (6.2)$$

where the prime denotes a derivative with respect to  $x$  and

$$f(u) = R^4 G_{tt} G_{xx} , \quad g(u) = G_{tt} G_{uu} . \quad (6.3)$$

We do not need to consider at all the contribution of the NSNS 2-form, since it is vanishing due to the ansatz (6.1) and to the fact that, for the extremal D3-brane distributions considered here, the 2-forms  $C_3^i$  generating terms for the NSNS 2-form through (2.8) vanish identically.

Since the action (6.2) does not explicitly depend on  $x$ , it leads to the first integral  $u_0$ , identified with the turning point of the solution. Solving the corresponding first-order equation for  $x$  in terms of  $u$  we find that the linear separation of the quark and antiquark is

$$L = 2R^2 f^{1/2}(u_0) \int_{u_0}^{\infty} du \sqrt{\frac{g(u)}{f(u)[f(u) - f(u_0)]}} . \quad (6.4)$$

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<sup>6</sup>If, for instance,  $\theta = \pi/2$ , then the resulting Wilson loop potentials are identical to those for the  $\mathcal{N} = 4$  undeformed theory computed in [42]. Also, to conform with standard notation in the literature, we use  $u$  instead of  $r$  in the Wilson-loop computations.

<sup>7</sup>In extending this to the finite-temperature case, one also has to take  $r_0 \rightarrow -ir_0$  due to the presence of nonzero metric components  $G_{ti} \sim r_0 dt d\phi_i$ .

The energy of the configuration is given by the action (6.2) divided by  $T$ . Subtracting the self-energy contribution, we obtain

$$E = \frac{1}{\pi} \int_{u_0}^{\infty} du \left[ \sqrt{\frac{g(u)f(u)}{f(u) - f(u_0)}} - \sqrt{g(u)} \right] - \frac{1}{\pi} \int_{u_{\min}}^{u_0} du \sqrt{g(u)} , \quad (6.5)$$

where  $u_{\min}$  is the minimum value of  $u$  allowed by the geometry. In specific examples, we are supposed to solve for the auxiliary parameter  $u_0$  in terms of the separation distance  $L$ . Since this cannot be done explicitly except for some special cases, in practice one regards Eq. (6.4) as a parametric equation for  $L$  in terms of the integration constant  $u_0$ . Combining it with Eq. (6.5) for  $E$ , one can then determine the behavior of the potential energy of the configuration in terms of the quark-antiquark separation.

## 6.2 Application: $\sigma$ -deformations of the Coulomb branch

As an application of the above, we extend the results of [42] for the undeformed theory, by considering the behavior of the static  $q\bar{q}$  potential for the case of pure  $\sigma$ -deformations ( $\hat{\gamma} = 0$ ) of the Coulomb branch of the gauge theory at zero temperature. This was previously examined in [16], where most of the qualitative features of the potentials were extracted numerically. Here, we pursue a more careful analysis, which allows us to discover certain features previously unnoticed.

For the analysis that follows, it is convenient to use the single dimensionful parameter  $r_0$  of the theory to switch to dimensionless variables. To do so, we set

$$u \rightarrow r_0 u , \quad u_0 \rightarrow r_0 u_0 , \quad (6.6)$$

and

$$L \rightarrow \frac{R^2}{r_0} L , \quad E \rightarrow \frac{r_0}{\pi} E . \quad (6.7)$$

Also, to keep the discussion at a reasonable length, we restrict to the cases of two equal rotation parameters and one rotation parameter. The results are presented below.

### Two equal rotation parameters (sphere)

For the case of two equal rotation parameters, the effective metric for the trajectories (6.1) reads

$$ds^2 = \mathcal{H}^{1/2} H^{-1/2} (-dt^2 + dx^2) + \mathcal{H}^{1/2} H^{1/2} du^2 , \quad (6.8)$$

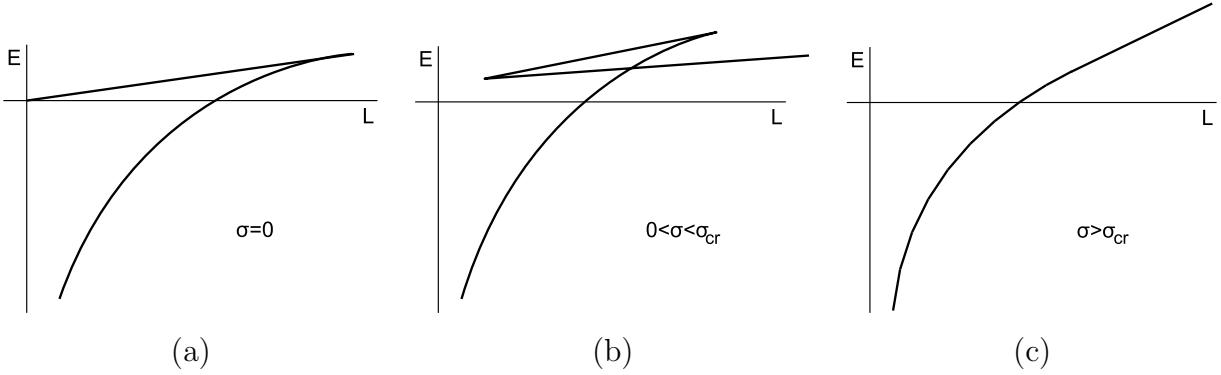


Figure 2: Energy as a function of length for the case of two equal rotation parameters.

with

$$\mathcal{H} = 1 + \frac{\hat{\sigma}^2 u^2}{u^2 - r_0^2}, \quad H = \frac{R^4}{u^2(u^2 - r_0^2)}. \quad (6.9)$$

Performing the analytic continuation, calculating the functions  $f$  and  $g$  according to (6.3), inserting into (6.4) and (6.5) and switching to dimensionless variables, we find the following exact expressions for  $L$  and  $E$  in terms of complete elliptic integrals<sup>8</sup>

$$\begin{aligned} L &= 2u_0 \sqrt{u_0^2 - \frac{1}{1 + \hat{\sigma}^2}} \int_{u_0}^{\infty} \frac{du}{u \sqrt{(u^2 - u_0^2)(u^2 - 1)(u^2 + u_0^2 - \frac{1}{1 + \hat{\sigma}^2})}} \\ &= \frac{2u_0}{\sqrt{(u_0^2 - \frac{1}{1 + \hat{\sigma}^2})(2u_0^2 - \frac{1}{1 + \hat{\sigma}^2})}} [\mathbf{P}(a^2, k) - \mathbf{K}(k)], \end{aligned} \quad (6.10)$$

and

$$\begin{aligned} E &= \sqrt{1 + \hat{\sigma}^2} \int_{u_0}^{\infty} du \sqrt{\frac{u^2 - \frac{1}{1 + \hat{\sigma}^2}}{u^2 - 1}} \left( \sqrt{\frac{u^2(u^2 - \frac{1}{1 + \hat{\sigma}^2})}{(u^2 - u_0^2)(u^2 + u_0^2 - \frac{1}{1 + \hat{\sigma}^2})}} - 1 \right) \\ &\quad - \sqrt{1 + \hat{\sigma}^2} \int_1^{u_0} du \sqrt{\frac{u^2 - \frac{1}{1 + \hat{\sigma}^2}}{u^2 - 1}} \\ &= \sqrt{1 + \hat{\sigma}^2} \left\{ \sqrt{2u_0^2 - \frac{1}{1 + \hat{\sigma}^2}} [\mathbf{a}^2 \mathbf{K}(k) - \mathbf{E}(k)] + \mathbf{E}(c) - \frac{\hat{\sigma}^2}{1 + \hat{\sigma}^2} \mathbf{K}(c) \right\}, \end{aligned} \quad (6.11)$$

where

$$k^2 = \frac{u_0^2 + \frac{\hat{\sigma}^2}{1 + \hat{\sigma}^2}}{2u_0^2 - \frac{1}{1 + \hat{\sigma}^2}}, \quad a^2 = \frac{u_0^2 - \frac{1}{1 + \hat{\sigma}^2}}{2u_0^2 - \frac{1}{1 + \hat{\sigma}^2}}, \quad c = \frac{1}{\sqrt{1 + \hat{\sigma}^2}}. \quad (6.12)$$

The resulting plots of  $E$  versus  $L$  are shown in Fig. 2. For  $\hat{\sigma} = 0$ , the behavior is that for the undeformed case [42]. As  $\hat{\sigma}$  is turned on, the length and energy curves closely resemble the van der Waals isotherms for a statistical system with  $u_0$ ,  $L$  and

<sup>8</sup>In the following we adopt the notation and use properties of elliptic integrals as in [43] and [44].

$E$  corresponding to volume, pressure and Gibbs potential respectively (see, for instance, [45]). In the region below a critical point,  $\hat{\sigma} < \hat{\sigma}_{\text{cr}}$ , the behavior is analogous to that of the statistical system at  $T < T_{\text{cr}}$ . Namely, the potential energy  $E$  (i) starts out Coulombic at small distances, (ii) becomes a triple-valued function of  $L$  with the state of lowest energy corresponding to the initial branch, (iii) passes a self-intersection point after which it becomes a triple-valued function of  $L$  with the state of lowest energy corresponding to the second branch and (iv) returns to being a single-valued function of  $L$  with approximately linear behavior. By standard arguments, the physical path in the length and energy curves must correspond to the physical isotherms of the statistical system, with the self-intersection point in the energy curve indicating a first-order phase transition with order parameter  $u_0$ . In the region above the critical point,  $\hat{\sigma} > \hat{\sigma}_{\text{cr}}$ , the behavior is analogous to that of the statistical system at  $T > T_{\text{cr}}$ . Now the energy is single-valued throughout and the first-order phase transition has degenerated into a second-order one between a Coulombic phase and a confining phase with a linear potential, as we will show below. The above critical behavior is similar to that found in [42] for a different system, namely for the undeformed theory at finite temperature and non-zero chemical potential. Since the two cases are in complete analogy, we refer the reader to that work for several related computational details.

To determine the critical value of  $\hat{\sigma}$ , we consider the derivative of the separation length  $L$  with respect to  $u_0$ . This derivative is proportional to the function

$$f(u_0; \hat{\sigma}) = \frac{1}{1 + 2\hat{\sigma}^2} \{ \hat{\sigma}^2 + 2(1 + \hat{\sigma}^2)u_0^2 - (1 + \hat{\sigma}^2)^2 u_0^2 [2u_0^2 \mathbf{K}(k) + (1 - u_0^2) \mathbf{E}(k)] \} . \quad (6.13)$$

This function has a single zero corresponding to a global maximum of the length for  $\hat{\sigma} = 0$ , two zeros corresponding to a local minimum and a local maximum of the length for  $0 < \hat{\sigma} < \hat{\sigma}_{\text{cr}}$ , and no extrema for  $\hat{\sigma} > \hat{\sigma}_{\text{cr}}$ . To calculate  $\hat{\sigma}_{\text{cr}}$ , we proceed by expanding  $f(u_0; \hat{\sigma})$  around  $u_0 = 1$  corresponding to the modulus  $k = 1$ ; although this procedure is not *a priori* valid, it will be justified by our final result. We have

$$f = -\hat{\sigma}^2 - 4x [8 + 15\hat{\sigma}^2 + (2 + \hat{\sigma}^2) \ln x] + \mathcal{O}(x^2) , \quad x \equiv \frac{1 + \hat{\sigma}^2}{8(1 + 2\hat{\sigma}^2)} (u_0 - 1) , \quad (6.14)$$

and

$$\frac{\partial f}{\partial x} = -4 [10 + 16\hat{\sigma}^2 + (2 + \hat{\sigma}^2) \ln x] + \mathcal{O}(x) . \quad (6.15)$$

Setting  $f = 0$  gives the transcendental equation

$$-a \ln a = \frac{\hat{\sigma}^2}{4(2 + \hat{\sigma}^2)} e^{\frac{8+15\hat{\sigma}^2}{2+\hat{\sigma}^2}} \leq e^{-1} , \quad a = x e^{\frac{8+15\hat{\sigma}^2}{2+\hat{\sigma}^2}} . \quad (6.16)$$

This equation has two solutions  $a_1, a_2$  if  $\hat{\sigma} < \hat{\sigma}_{\text{cr}}$ , one solution  $a_1 = a_2 = a_c$  if  $\hat{\sigma} = \hat{\sigma}_{\text{cr}}$  and no solutions if  $\hat{\sigma} > \hat{\sigma}_{\text{cr}}$ . The critical value  $\hat{\sigma}_c$  is obtained when the above inequality is saturated in which case the solution is  $a = e^{-1}$ . It turns out that in this case we have in addition that  $\partial f / \partial x = 0$ . The corresponding transcendental equation is

$$ze^{z+5} = 44 , \quad z \equiv 11 \frac{\hat{\sigma}_{\text{cr}}^2}{2 + \hat{\sigma}_{\text{cr}}^2} . \quad (6.17)$$

This gives  $z \simeq 0.235$ , whence  $\hat{\sigma}_{\text{cr}} \simeq 0.209$ , which clearly is small enough to validate in retrospect the approximation method we used to compute it. Having a solution to (6.16) we obtain from the definition in (6.14) that the corresponding critical value(s) for  $u_0$  are given by

$$u_{0i} = 1 + \frac{a_i}{8} \frac{1 + 2\hat{\sigma}^2}{1 + \hat{\sigma}^2} e^{-\frac{8+15\hat{\sigma}^2}{2+\hat{\sigma}^2}} , \quad i = 1, 2 , \quad (6.18)$$

for  $\hat{\sigma} \leq \hat{\sigma}_{\text{cr}}$ .

Finally, note that for  $u_0 \rightarrow 1$  (large  $L$ ) we recover the usual Coulombic behavior, while for  $u_0 \rightarrow 1$  we find the asymptotics

$$L \simeq \frac{\hat{\sigma}}{\sqrt{1 + 2\hat{\sigma}^2}} \ln \frac{1}{u_0 - 1} , \quad E \simeq \frac{\hat{\sigma}^2}{2\sqrt{1 + 2\hat{\sigma}^2}} \ln \frac{1}{u_0 - 1} . \quad (6.19)$$

Combining these expressions, we obtain the potential

$$E \simeq \frac{\hat{\sigma}}{2} L , \quad (6.20)$$

which demonstrates the linear confining behavior claimed earlier on, as long as  $\hat{\sigma} > 0$ . This linear potential was also found in the studies of [16] where, however, the critical behavior found here was missed. Using the first of (6.19) and reinstating dimensional units according to the first (6.7), we find that confinement sets in at length scales  $L \gtrsim \frac{\hat{\sigma}}{\sqrt{1+2\hat{\sigma}^2}} \frac{R^2}{r_0}$ .

### One rotation parameter (disc)

For the case of one rotation parameter, the effective metric for the trajectories (6.1) reads

$$ds^2 = \mathcal{H}^{1/2} H^{-1/2} (-dt^2 + dx^2) + \mathcal{H}^{1/2} H^{1/2} du^2 , \quad (6.21)$$

with

$$\mathcal{H} = 1 + \frac{\hat{\sigma}^2 u^2}{u^2 + r_0^2} , \quad H = \frac{R^4}{u^2(u^2 + r_0^2)} . \quad (6.22)$$

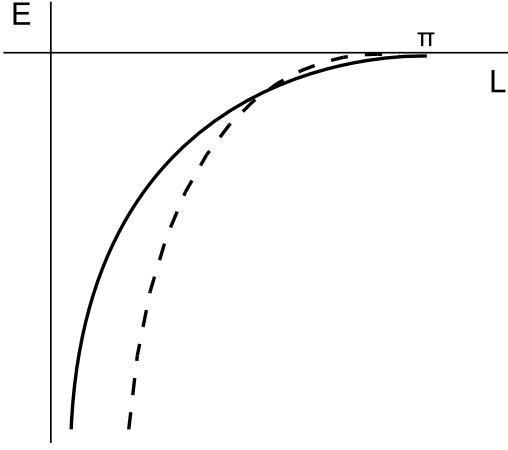


Figure 3: Energy as a function of length for the case of one rotation parameter, shown for  $\hat{\sigma} = 0$  (solid) and  $\hat{\sigma} > 0$  (dashed).

Proceeding in the same way as before, we find that the length and potential energy are given by

$$\begin{aligned}
 L &= 2u_0 \sqrt{u_0^2 + \frac{1}{1+\hat{\sigma}^2}} \int_{u_0}^{\infty} \frac{du}{u \sqrt{(u^2 - u_0^2)(u^2 + 1)(u^2 + u_0^2 + \frac{1}{1+\hat{\sigma}^2})}} \\
 &= \frac{2u_0}{\sqrt{(u_0^2 + \frac{1}{1+\hat{\sigma}^2})(2u_0^2 + \frac{1}{1+\hat{\sigma}^2})}} [\mathbf{\Pi}(a^2, k) - \mathbf{K}(k)] , \tag{6.23}
 \end{aligned}$$

and

$$\begin{aligned}
 E &= \sqrt{1+\hat{\sigma}^2} \int_{u_0}^{\infty} du \sqrt{\frac{u^2 + \frac{1}{1+\hat{\sigma}^2}}{u^2 + 1}} \left( \sqrt{\frac{u^2(u^2 + \frac{1}{1+\hat{\sigma}^2})}{(u^2 - u_0^2)(u^2 + u_0^2 + \frac{1}{1+\hat{\sigma}^2})}} - 1 \right) \\
 &\quad - \sqrt{1+\hat{\sigma}^2} \int_0^{u_0} du \sqrt{\frac{u^2 + \frac{1}{1+\hat{\sigma}^2}}{u^2 + 1}} \\
 &= \sqrt{1+\hat{\sigma}^2} \left\{ \sqrt{2u_0^2 + \frac{1}{1+\hat{\sigma}^2}} [a^2 \mathbf{K}(k) - \mathbf{E}(k)] + \mathbf{E}(c) - \frac{1}{1+\hat{\sigma}^2} \mathbf{K}(c) \right\} . \tag{6.24}
 \end{aligned}$$

where

$$k^2 = \frac{u_0^2 - \frac{\hat{\sigma}^2}{1+\hat{\sigma}^2}}{2u_0^2 + \frac{1}{1+\hat{\sigma}^2}} , \quad a^2 = \frac{u_0^2 + \frac{1}{1+\hat{\sigma}^2}}{2u_0^2 + \frac{1}{1+\hat{\sigma}^2}} \quad c = \frac{\hat{\sigma}}{\sqrt{1+\hat{\sigma}^2}} . \tag{6.25}$$

Fig. 3 shows the resulting plots of  $E$  versus  $L$ . For  $u_0 \gg 1$  (small  $L$ ) we recover the standard Coulombic behavior [15] enhanced by the factor  $\sqrt{1+\hat{\sigma}^2}$ , while for  $u_0 \rightarrow 0$  we find the asymptotics

$$L \simeq \pi - 2\mathbf{E}(i\hat{\sigma})u_0 , \quad E \simeq -\frac{1}{2}\mathbf{E}(i\hat{\sigma})u_0^2 , \tag{6.26}$$

which reduce to those in [42] for  $\hat{\sigma} = 0$ . Combining these expressions, we obtain

$$E \simeq -\frac{(\pi - L)^2}{8\mathbf{E}(i\hat{\sigma})}, \quad (6.27)$$

which shows that there is complete screening at the screening length  $L_c = \pi$  that is invariant under  $\sigma$ -deformations and the same as the one found [42] for the undeformed case. The effect of  $\sigma$ -deformations is to enhance the quark-antiquark force for small separations and to suppress it for separations close to the screening length.<sup>9</sup> All of our results reduce smoothly to those in [42] for  $\hat{\sigma} = 0$ .

We finally remark that it would be very interesting to examine the stability of the string trajectories used for calculating the quark-antiquark potential in the deformed theories. For instance, in the undeformed theory with two rotation parameters, the potential for our trajectory (6.1) with  $\theta = 0$  (shown in Fig. 2(a)) is a double-valued function while the potential for a trajectory with  $\theta = \pi/2$  (which is insensitive to deformations and thus has not been considered here) exhibits a confining behavior at large distances [42]. It was found in [46] that the upper branch in the  $\theta = 0$  case as well as the region giving rise to linear behavior in the  $\theta = \pi/2$  case are actually unstable under small fluctuations, the latter fact being in accordance with our physical expectation about the absence of confinement in  $\mathcal{N} = 4$  SYM. In the presence of deformation, the potential for  $\theta = 0$  gives the behavior shown in Fig. 2(b,c), while the potential for  $\theta = \pi/2$  stays invariant. It is then important to ask whether the confining regions of these potentials are stable, as the  $\mathcal{N} = 1$  supersymmetry of the  $\sigma$ -deformed theories actually leads us to expect a confining behavior at large distances, at least in some regions of the moduli space. We note that the question of stability is meaningful even in the  $\theta = \pi/2$  case since, although the classical string solution is independent of the deformation, small fluctuations about it are *not*. We hope to report on work in that direction in the future [47].

## 7 Conclusions

In this paper, we have explicitly applied the Lunin–Maldacena construction of complex marginal deformations of supergravity solutions to a class of general Type IIB backgrounds that include the gravity duals of  $\mathcal{N} = 4$  gauge theories at finite temperature

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<sup>9</sup>Note the crossover behavior for the curves with  $\hat{\sigma} = 0$  and  $\hat{\sigma} > 0$  in our Fig. 3 at a certain length. This is understood by our analytic result (6.26) as well as the enhanced Coulombic behavior in the UV noted above. In that respect we disagree with the shape of the potential presented in Fig. 3 of [16].

and R-charge chemical potentials and at the Coulomb branch. For these theories, we have concentrated on three simple cases of the general solution, we have presented in full detail the marginally-deformed metrics, and we have checked that their thermodynamics (for the rotating case) are the same as for the undeformed ones, as they should.

Having constructed the marginally-deformed spacetimes, we considered their Penrose limits for the multicenter case along a certain class of geodesics inside the angular part of the geometry. Besides recovering familiar results, we have extended them to take account of the presence of  $\sigma$ -deformations, which has not been considered in the literature up to date (with the exception of the archetypal example of [3]), as well as for the presence of rotation (previously examined in [27] for the undeformed solutions). We have also considered Penrose limits along a non-BPS geodesic.

We next turned to a study of the giant gravitons supported on the PP-wave spacetimes just constructed, using the simple example of the PP-wave of [3] but taking the  $\sigma$ -deformation into account. For that case we found that, unlike  $\gamma$ -deformations,  $\sigma$ -deformations lift the degeneracy between the giant and the point graviton but, nevertheless, the former remains stable under small perturbations. We also showed that, for the particular geodesic under consideration, this result is unaffected by the presence of rotation. We also considered dual giant gravitons, in which case the situation remains qualitatively unchanged by the deformation. It would be interesting to generalize these studies to more complicated PP-waves and to seek giant graviton solutions in the full  $\sigma$ -deformed geometry.

Finally, we considered the standard Wilson-loop calculation of interquark potentials and screening lengths for the dual gauge theory. Here we have considered the static heavy-quark potential in the case of the Coulomb branch of the  $\sigma$ -deformed gauge theory, and for the two D3-brane distributions under consideration we found a linear confining potential and a screened Coulombic potential respectively. For the first case, we elucidated on the nature of the transition to the confining phase and we calculated the critical value of the deformation parameter, while for the second case we demonstrated invariance of the screening length under the deformation. As noted in the text, it is very interesting to examine the stability of the string trajectories used to calculate potentials in the deformed theory using the formalism developed in [46]. Also it is of some interest to extend these results to the theory at finite temperature and chemical potential.

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## A T– and S–duality rules

Here, we state our conventions for the T– and S–duality transformations used in Section 2. Starting from T–duality, we consider a Type II configuration characterized by the metric  $G_{MN}$ , the NSNS 2-form  $B_2$ , the dilaton  $\Phi$  and the RR  $p$ -forms  $A_p$  and we wish to T–dualize along an isometry direction, say  $y$ . Splitting the coordinates as  $x^M = (x^\mu, y)$ , we decompose the metric, the NSNS 2-form and the RR  $p$ -forms into the quantities

$$a_1 \equiv G_{y\mu} dx^\mu, \quad \phi \equiv G_{yy}, \quad b_2 \equiv \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu, \quad b_1 \equiv B_{y\mu} dx^\mu, \quad (\text{A.1})$$

and

$$\alpha_p \equiv \frac{1}{p!} A_{\mu \dots \nu \rho} dx^\mu \wedge \dots \wedge dx^\nu \wedge dx^\rho, \quad \beta_{p-1} \equiv \frac{1}{(p-1)!} A_{\mu \dots \nu y} dx^\mu \wedge \dots \wedge dx^\nu, \quad (\text{A.2})$$

so that, in particular,  $B_2 = b_2 - b_1 \wedge dy$  and  $A_p = \alpha_p + \beta_{p-1} \wedge dy$ . Under T–duality, the NSNS fields transform among themselves by the usual Buscher rules [48] while the RR fields transform by terms involving both NSNS and RR fields [49]. In our present notation, these transformations can be written in the compact form

$$\begin{aligned} \hat{G}_{MN} dx^M dx^N &= G_{\mu\nu} dx^\mu dx^\nu + \phi^{-1} [(dy + b_1)^2 - a_1^2] \\ \hat{B}_2 &= b_2 - \phi^{-1} a_1 \wedge (dy + b_1), \\ e^{2\hat{\Phi}} &= \phi^{-1} e^{2\Phi}, \\ \hat{A}_p &= \beta_p + (\alpha_{p-1} - \phi^{-1} \beta_{p-2} \wedge a_1) \wedge (dy + b_1). \end{aligned} \quad (\text{A.3})$$

This form of the T–duality rules is particularly useful in the computations of section 2, as it allows us to perform the transformations directly in form notation.

We next consider an S–duality transformation of a Type IIB configuration, which acts on the axion-dilaton  $\tau = A_0 + i e^{-\Phi}$  and the NSNS and RR 2-forms as follows

$$\hat{\tau} = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} -\hat{B}_2 \\ \hat{A}_2 \end{pmatrix} = (\Lambda^T)^{-1} \begin{pmatrix} -B_2 \\ A_2 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (\text{A.4})$$

leaving the Einstein-frame metric and the RR 5-form field strength invariant. For the purpose of  $\sigma$ -deformations, we consider the particular  $SL(2, \mathbb{R})$  element

$$\Lambda = \begin{pmatrix} 1 & 0 \\ -\tilde{\sigma} & 1 \end{pmatrix} . \quad (\text{A.5})$$

which, in addition, leaves the RR 2-form potential  $A_2$  invariant. The resulting transformations of all fields, including the string-frame metric and the RR 4-form, are written explicitly as

$$\begin{aligned} \hat{G}_{MN} &= \lambda^{1/2} G_{MN} , \\ \hat{B}_2 &= B_2 - \tilde{\sigma} A_2 , \\ e^{2\hat{\Phi}} &= \lambda^2 e^{2\Phi} , \\ \hat{A}_0 &= \lambda^{-1} [A_0(1 - \tilde{\sigma} A_0) - \tilde{\sigma} e^{-2\Phi}] , \\ \hat{A}_4 &= A_4 - \frac{1}{2} \tilde{\sigma} A_2 \wedge A_2 . \end{aligned} \quad (\text{A.6})$$

where

$$\lambda \equiv (1 - \tilde{\sigma} A_0)^2 + \tilde{\sigma}^2 e^{-2\Phi} . \quad (\text{A.7})$$

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